

# Multi-Agent Systems

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## Motivation



The famous prisoner's dilemma with the following **payoff matrix**:

	<i>Silent</i>	<i>Betray</i>
<i>Silent</i>	-1, -1	-3, 0
<i>Betray</i>	0, -3	-2, -2

In games like this one cooperation is prevented, because:

- Binding agreements are not possible
- Pay-off is given directly to individuals as the result of individual action

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## Cooperative Game Theory



- In many situations...
  - **Contracts** can form binding agreements
  - Pay-off is given to **groups** of agents rather than to individuals
- Hence, cooperation is both possible and rational.
- **Cooperative game theory** asks which contracts are meaningful solutions among self-interested agents.

**Characterization (Shoham, Keyton-Brown, 2009, Ch. 12)**

[Cooperative game theory is about] how self-interested agents can combine to form effective teams.

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## Real-world examples



- Political parties form coalitions to ensure majorities. Division of power (ministry posts).
- Companies cooperate to save resources.
- People buy expensive things together they could not afford to buy alone.
- Buildings are built by several people with different capabilities (craftsmen, electricians, architects, ...). Who should earn how much?

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- Which coalition should/will form?
- How should the value be divided among the members?

### Cooperative Game (with transferable utility)

A cooperative game with transferable utility is a pair  $(N, v)$ :

- $N$ : Set of agents
- Any subset  $S \subseteq N$  is called a **coalition**
- $N$  is the **grand coalition**
- $v : 2^N \rightarrow \mathbb{R}$ : characteristic function that assigns a value  $v(S)$  to each  $S \subseteq N$ ,  $v(\emptyset) = 0$ , also called the **payoff of  $S$** .
- Transferable value assumption:
  - Value of a coalition can be (arbitrarily) redistributed among the coalition's members
  - I.e., value is dispensed in some universal currency
  - Each coalition can be assigned a single value

## Division of Value

- $\Psi(S, v) = (\Psi_1(S, v), \dots, \Psi_k(S, v))$  is a **distribution of value** to members  $1, \dots, k$  of  $S$ .

### Feasible distribution

A distribution  $\Psi(S, v)$  is **feasible** iff

$$\sum_{i \in N} \Psi_i(N, v) \leq v(N)$$

### Efficient distribution

A distribution  $\Psi(S, v)$  is **efficient** iff

$$\sum_{i \in N} \Psi_i(N, v) \geq v(N)$$

## Example: Gloves

Mr A and Mr B are knitting gloves. The gloves are one-size-fits-all, and two gloves make a pair that they sell for 5 EUR. They have each made three gloves. How to share the proceeds from the sale?

## Example: Gloves



Mr A and Mr B are knitting gloves. The gloves are one-size-fits-all, and two gloves make a pair that they sell for 5 EUR. They have each made three gloves. How to share the proceeds from the sale?

- $(N, v), N = 1, 2, v(\{1\}) = 5, v(\{2\}) = 5, v(\{1, 2\}) = 15$
- Assume they form a coalition. What about these feasible and efficient divisions of value?
  - $\Psi_a = (7.5, 7.5)$
  - $\Psi_b = (5, 10)$
  - $\Psi_c = (4, 11)$

## Example: Gloves Extended



Mr A and Mr B and Mr C are knitting gloves. The gloves are one-size-fits-all, and two gloves make a pair that they sell for 5 EUR. They have each made three gloves. How to share the proceeds from the sale?

- $(N, v), N = 1, 2, 3$ 
  - $v(\{1\}) = 5, v(\{2\}) = 5, v(\{3\}) = 5, v(\{1, 2\}) = 15, v(\{1, 3\}) = 15, v(\{2, 3\}) = 15, v(\{1, 2, 3\}) = 20$
- Assume they form a coalition. What about these feasible and efficient divisions of value?
  - $\Psi_d = (6.6, 6.6, 6.6)$
  - $\Psi_e = (7.5, 7.5, 5)$

## The Core: Definition



### Core

The **core** of a cooperative game  $(N, v)$  is the set of feasible and efficient distributions of value  $\Psi$ , such that no  $S \subseteq N$  can do better by splitting off, i.e.,

$$\sum_{i \in S} \psi_i \geq v(S)$$

## Existence of the core



- Is the core always nonempty? No.
  - In the extended glove example, the core is empty.

- Is the core always unique? No.
  - In the original glove example, all  $PS_i = (\Psi_1, \Psi_2)$  such that  $\Psi_1 > 5, \Psi_2 > 5, \Psi_1 + \Psi_2 = 15$  are in the core.

### Simple game

A game  $(N, v)$  is a **simple game** iff for all  $S \subseteq N, v(S) \in \{0, 1\}$

### Veto agent

An agent  $i$  is a **veto agent** iff  $v(N \setminus \{i\}) = 0$ .

Consider the coalition of three parties A, B, C with 30, 25, and 15 votes, respectively.

- Case 55 votes necessary to win the election.
  - $v(A) = 0, v(B) = 0, v(AB) = 1, v(AC) = 0, v(BC) = 0, v(ABC) = 1$
  - Who is a veto agent? How does the core look like?
- Case 15 votes necessary to win the election.
  - $v(A) = 1, v(B) = 1, v(AB) = 1, v(AC) = 1, v(BC) = 1, v(ABC) = 1$
  - Who is a veto agent? How does the core look like?

### Theorem

In a simple game the core is empty iff there is no veto agent. If there are veto agents, the core consists of all value distributions in which the nonveto agents get 0.

- **Goal:** Coalition is to divide its value 'fair'.
- **Shapley's idea:** Members should receive value proportional to their contributions.
- **However:**
  - Consider  $v(N) = 1$  and  $v(S) = 0$  for all  $S \neq N$ .
  - Thus,  $v(N) - v(N \setminus \{i\}) = 1$  for every agent  $i$ : everybody's contribution is 1 (everybody is indeed likewise essential).
  - Clearly, one cannot pay 1 to everybody
  - Needed: Some way of weighing. How to design it?
  - Next: **Axiomatic characterization** of properties of a fair value division (due to Shapley).

### Definition Interchangeability

Agents  $i$  and  $j$  are **interchangeable** relative to  $v$  iff they always contribute the same amount to every coalition of the other agents, i.e., for all  $S$  that contain neither  $i$  nor  $j$ ,  
 $v(S \cup \{i\}) = v(S \cup \{j\})$ .

### Axiom Symmetry

For any  $S \subseteq N, v$ , if  $i$  and  $j$  are interchangeable then  
 $\Psi_i(S, v) = \Psi_j(S, v)$ .

- Agents who contribute the same to every possible coalition should get the same.

### Definition Dummy Player

Agent  $i$  is a **dummy player** iff the amount that  $i$  contributes to any coalition is  $v(\{i\})$ , i.e., for all  $S \setminus \{i\}$ ,  $v(S \cup \{i\}) = v(S) + v(\{i\})$ .  
If  $v(\{i\}) = 0$ ,  $i$  is called a **null player**.

### Axiom Dummy Player

For any  $S \subseteq N, v$  if  $i$  is a dummy player then  $\Psi_i(S, v) = v(\{i\})$ .

### Axiom Additivity

For any two  $v_1, v_2$ , it holds that  
 $\Psi_i(N, v_1 + v_2) = \Psi_i(N, v_1) + \Psi_i(N, v_2)$  for each  $i$ , where the game  $(N, v_1 + v_2)$  is defined by  $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ .

## Theorem

Given a coalitional game  $(N, v)$ , there is a unique payoff division  $\Psi(N, v)$  that divides the full payoff of the grand coalition and that satisfies Symmetry, Dummy Player, and Additivity: The **Shapley Value**.

## Marginal value of agent $i$

The **marginal value of an agent  $i$**  to any coalition  $S \subseteq N$  is defined by  $\mu_i : 2^N \rightarrow \mathbb{R}$ :

$$\mu_i(S) := \begin{cases} v(S \cup \{i\}) - v(S), & i \notin S \\ v(S) - v(S \setminus \{i\}), & i \in S \end{cases}.$$

## Definition Shapley Value

Given a cooperative game  $(N, v)$ , the **Shapley Value** divides value according to:

$$\Psi_i(N, v) = \frac{1}{N!} \sum_{o \in \Pi(N)} \mu_i(C_i(o))$$

- $\Pi_n = \{(x_1, \dots, x_n) \mid x_i \in N, \forall i, j [i \neq j \Rightarrow x_i \neq x_j]\}$
- $C_i(o)$ : set containing only those agents that appear before agent  $i$  in  $o$ , e.g.,  $o = (3, 1, 2)$ , then  $C_3(o) = \emptyset$ ,  $C_2(o) = \{1, 3\}$

- Original Glove Example: Shapley-Value (7.5, 7.5)
  - Permutation AB
    - Marginal Contribution of A: 5
    - Marginal Contribution of B: 10
  - Permutation BA
    - Marginal Contribution of A: 10
    - Marginal Contribution of B: 5

## Examples: Shapley Value

- Extended Glove Example: Shapley-Value (6.6, 6.6, 6.6)
  - Permutation ABC
    - Marginal Contributions of A, B, C: 5, 10, 5
  - Permutation ACB
    - Marginal Contributions of A, B, C: 5, 5, 10
  - Permutation BAC
    - Marginal Contributions of A, B, C: 10, 5, 5
  - Permutation BCA
    - Marginal Contributions of A, B, C: 5, 5, 10
  - Permutation CBA
    - Marginal Contributions of A, B, C: 5, 10, 5
  - Permutation CAB
    - Marginal Contributions of A, B, C: 10, 5, 5

⇒ Not in the core!

## Convex Games

### Convex game

A game  $(N, v)$  is **convex**, iff the value of a coalition increases no slower when these coalitions grow in size, i.e.,  
 $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$  for all  $S \subseteq T \subseteq N, i \in N \setminus T$ .

### Theorem

In every convex game, the Shapley value is in the core.

### Theorem

Every convex game has a nonempty core.

⇒ Fair and stable distributions exist!

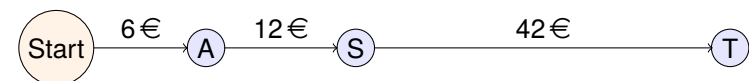
## Interesting Example I: Bankruptcy game instance

- Claimants:  $N = \{A, B\}$
- Claims:  $c_A = 80, c_B = 40$
- Estate:  $E = 100$
- $v(C) = \max\{0, \sum_{i \in N \setminus C} c_i\}$ 
  - $v(\emptyset) = 0, v(\{A\}) = 60, v(\{B\}) = 20, v(\{A, B\}) = 100$

### Properties

- This game is convex ⇒ the Shapley value is in the core.
- Shapley value:  $\Psi = (\Psi_A, \Psi_B) = \frac{(60, 40) + (80, 20)}{2} = (70, 30)$
- In core indeed, because:
  - $\Psi_A = 70 \geq v(\{A\}) = 60$  ☺
  - $\Psi_B = 30 \geq v(\{B\}) = 20$  ☺
  - $\Psi_A + \Psi_B = 70 + 30 \geq v(\{A, B\}) = 100$  ☺

## Interesting Example II: Taxi Share



### ■ characteristic function $v$

- $v(\{A\}) = 6$
- $v(\{S\}) = 12$
- $v(\{T\}) = 42$
- $v(\{A, S\}) = 12$
- $v(\{A, T\}) = 42$
- $v(\{S, T\}) = 42$
- $v(\{A, S, T\}) = 42$

### ■ Shapley value computation

- $(A, S, T) \rightarrow (6, 6, 30)$
- $(A, T, S) \rightarrow (6, 0, 36)$
- $(S, A, T) \rightarrow (0, 12, 30)$
- $(S, T, A) \rightarrow (0, 12, 30)$
- $(T, A, S) \rightarrow (0, 0, 42)$
- $(T, S, A) \rightarrow (0, 0, 42)$
- $\Psi(N, v) = (2, 5, 35)$

- Cooperative game theory is concerned with what agents can achieve if they form **coalitions**, viz., binding agreements.
  - Values are given to coalitions first
  - Coalitions redistribute value to their members
- Solution concepts for cooperative games
  - Core: stability; sometimes exists; not unique
  - Shapley value: fairness; always exists; unique
  - For convex games, the Shapley value is in the Core
- **Next**
  - Computational aspects
  - Coalition structure formation

Remember: Given a cooperative game  $(N, v)$ , the **Shapley Value** divides value according to:

$$\Psi_i(N, v) = \frac{1}{N!} \sum_{o \in \Pi(N)} \mu_i(C_i(o))$$

- Imagine you wanted to compute the Shapley value of an agent  $i$  of a cooperative game  $(N, v)$

**def** shapleyValue( $N, v, i$ ):

...

- How many entries are in  $v$ ?
- How many steps are necessary to compute Shapley value?

- Some cooperative games can be treated more efficiently
  - Weighted graph games
  - Weighted voting games
- Centralized algorithm for coalition structure generation

### Assumption

The value of a coalition is the sum of the pairwise synergies among agents.

### Definition

Let  $(V, W)$  denote an undirected weighted graph, where  $V$  is the set of vertices and  $W \in \mathbb{R}^{V \times V}$  is the set of edge weights; denote the weight of the edge between vertices  $i$  and  $j$  as  $w_{\{i,j\}}$ . This graph defines a weighted graph game, where the cooperative game is constructed as follows:

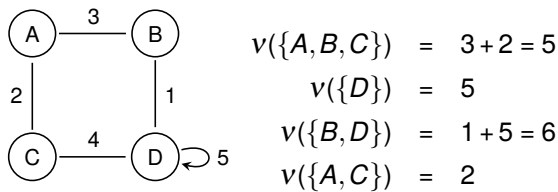
- $N = V$
- $v(S) = \sum_{\{i,j\} \subseteq S} w_{\{i,j\}}$



## Weighted graph game: Example

### Revenue Sharing game

Consider the problem of dividing the revenues from toll highways between the cities that the highways connect. The pair of cities connected by a highway get to share in the revenues only when they form an agreement on revenue splitting; otherwise, the tolls go to the state.



## Weighted graph game: Shapley Value

- 1 Only  $N^2$  many values to store (adjacency matrix).
- 2 Shapley-Value  $sh_i$  of agent  $i$ :  $sh_i = w_{\{i,i\}} + \frac{1}{2} \sum_{i \neq j} w_{\{i,j\}}$

Each pair of agents plays a game, in which they are interchangeable. Thus, they get the same value (**Symmetry**).

### Axiom Symmetry

For any  $S \subseteq N, v$ , if  $i$  and  $j$  are interchangeable then  $\Psi_i(S, v) = \Psi_j(S, v)$ .

Value adds up in “bigger” games due to **Additivity**.

### Axiom Additivity

For any two  $v_1, v_2$ , it holds that

$\Psi_i(N, v_1 + v_2) = \Psi_i(N, v_1) + \Psi_i(N, v_2)$  for each  $i$ , where the game  $(N, v_1 + v_2)$  is defined by  $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ .

## Weighted graph game: Properties

### Theorem

If all the weights are nonnegative then the game is convex.

Remember:

### Theorem

Every convex game has a nonempty core.

### Theorem

In every convex game, the Shapley value is in the core.

⇒ A fair and stable value distribution exists and can be computed in polynomial time w.r.t. to number of agents.

## Weighted graph game: Incompleteness

- Not every game can be represented as a weighted graph game. For example, consider this voting game:
- Parties A, B, C, D have 45, 25, 15, and 15 representatives, respectively. 51 votes are required to pass the bill.
  - Agents:  $N = \{A, B, C, D\}$
  - Coalitions:  $\{A\}, \dots, \{A, B, C, D\} \in 2^N$
  - Characteristic function  $v : 2^N \rightarrow \mathbb{R}$ 
    - $v(\{A\}) = v(\{B\}) = v(\{C\}) = v(\{D\}) = v(\{B, C\}) = v(\{B, D\}) = v(\{C, D\}) = 0$
    - $v(\{A, B\}) = v(\{A, C\}) = v(\{A, D\}) = v(\{B, C, D\}) = 1$
  - E.g.,  $v(\{B, C\}) + v(\{B, D\}) + v(\{C, D\}) \neq v(\{B, C, D\})$  ☹, i.e., the overall payoff is not the result of the local coalitions.

### Definition

A weighted voting game  $(q; w_1, \dots, w_n)$  consists of a set of agents  $Ag = \{1, \dots, n\}$  and a quota  $q$ . The cooperative game  $(N, v)$  is then given by:

- $N = Ag$
- $v(C) = \begin{cases} 1, & \sum_{i \in C} w_i \geq q \\ 0, & \text{else} \end{cases}$

- Parties A, B, C, D have 45, 25, 15, and 15 representatives, respectively. 51 votes are required to pass the bill.
  - Weighted voting game:  $(51; 45, 25, 15, 15)$

- Computing the Shapley value is NP-hard
- But checking if core is non-empty is easy

Remember:

### Theorem

In a simple game the core is empty iff there is no veto agent. If there are veto agents, the core consists of all value distributions in which the nonveto agents get 0.

- Check if agent  $i$  is veto agent:
  - 1 Draw up  $C = N \setminus \{i\}$
  - 2 Check that both hold:
    - $\sum_{j \in C} w_j < q$ , i.e., no winner without  $i$
    - $\sum_{j \in C \cup \{i\}} w_j \geq q$ , i.e., winner with  $i$

- Some cooperative games can be treated more efficiently
  - Weighted graph games
  - Weighted voting games
- Centralized algorithm for coalition structure generation

- Agents can use their capacity to compute Shapley values to try to optimize their local payoff.
- If, however, there is a central component that knows of all the agents, this component can attempt to **maximize social welfare** of the whole system.

A **coalition structure** is a **partition** of the overall set of agents  $N$  into **mutually disjoint coalitions**.

Example, with  $N = \{1, 2, 3\}$ :

- Seven possible coalitions:

$\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}$

- Five possible coalition structures:

$\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\},$

$\{\{3\}, \{1, 2\}\}, \{\{1, 2, 3\}\}$

Given game  $G = (N, v)$ , the **socially optimal coalition structure**  $CS^*$  is defined as:

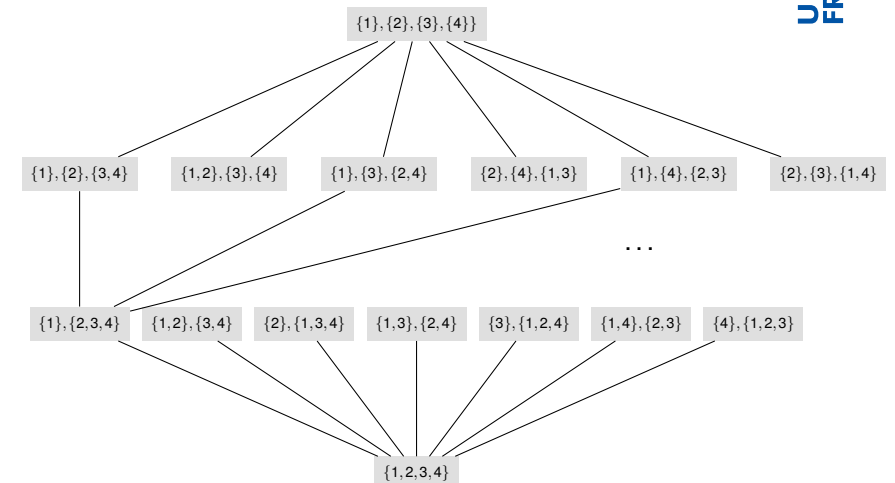
$$CS^* = \operatorname{argmax}_{CS \in \text{partitions of } N} V(CS)$$

where

$$V(CS) = \sum_{C \in CS} v(C)$$

Unfortunately, there are **exponentially many** coalition structures over the sets of agents  $N$

⇒ **Exhaustive search is infeasible!**



- Observation: At the first two levels every coalition is present.
- Let  $CS'$  be the best structure we find in these levels.
- Let  $CS^*$  be the best structure overall (as defined earlier).
- Let  $C^* = \operatorname{argmax}_{C \subseteq N} v(C)$  the coalition with highest possible value.




Then:

- $V(CS^*) \leq |N|v(C^*) \leq |N|V(CS')$
- $\Rightarrow$  in worst case,  $V(CS') = \frac{V(CS^*)}{|N|}$

Algorithm:

- 1 Search first two bottom levels, keep track of best one.
- 2 Continue with breadth-first search beginning with top level.

- The Core: Stability / rationality of a coalition
- Shapley Value: Fairness
- Simple Games: Core easy to determine
- Convex Games: Shapley Value is in the core
- Weighted graph games
  - Compact representation of games with additive values
  - Efficient computation of Shapley values
- Weighted voting games
  - Compact representation of certain simple games
  - Efficient computation of a core value distribution
- Coalition Structure Formation
  - Centralized search-based algorithm to find a partition of agents into coalitions maximizing overall value.
  - Provable bounds of solution quality.

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