Multi-Agent Systems

Albert-Ludwigs-Universität Freiburg







- Last session: Axioms of epistemic/doxastic logic, group knowledge (common knowledge, distributed knowledge).
- Today: Modeling changes of knowledge due to public communication and observations (muddy children puzzle).



Consider *n* children playing outdoors together. Suppose *k* of them get mud on their foreheads. Each of the *n* children can see which of the other n - 1 children are muddy or not, but, of course, can't be sure whether s/he is muddy.

- The father shows up and announces: "At least one of you has mud on his/her forehead."
- The father then asks: "Does any of you know whether s/he has mud on her/his forehead?"
- 3 After the *k*-th such question, all the *k* muddy children will answer "Yes!".



- Did the father tell the children anything new in the first announcement?
- Why is it that all the muddy children simultaneously know the answer to question (2) after exactly *k* rounds?



Case k = 1

- The muddy child only sees clean children. And all clean children see one muddy child.
 - Muddy child considers possible: 0 or 1 children are muddy.
 - Clean children consider possible: 1 or 2 children are muddy.
- After the father announces that at least one of them is muddy:
 - Muddy child considers possible: 1 muddy.
 - Clean children consider possible: 1 or 2 muddy.
- The father asks who knows to be muddy:
 - Muddy child knows!

Case *k* = 2

- The muddy children see exactly one muddy child. And all clean children see two muddy children.
 - Muddy children consider possible: 1 or 2 children are muddy.
 - Clean children consider possible: 2 or 3 children are muddy.
- After the father announces that at least one of them is muddy:
 - Muddy children consider possible: 1 or 2 muddy.
 - Clean children consider possible: 2 or 3 muddy.
- The father asks who knows to be muddy:
 - Nobody!
- Hence, there must be more than one muddy children.
 - Muddy children consider possible: 2 muddy.
 - Clean children consider possible: 2 or 3 muddy.
- The father asks who knows to be muddy:
 - The muddy children know!

Muddy Children: Initial

(reflexive edges omitted)



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 $\begin{array}{c} M, w_{1} \models \\ C_{\{1,2\}}(K_{1}m_{2} \lor K_{1} \neg m_{2}) \\ M, w_{1} \models \\ C_{\{1,2\}}(K_{2}m_{1} \lor K_{2} \neg m_{1}) \\ \end{array} \begin{array}{c} M, w_{1} \models \\ M$

Muddy Children: After First Announcement (reflexive edges omitted)

Father:"At least one of you has mud on his/her forehead!"





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Muddy Children: After Question

(reflexive edges omitted)

Nobody answers "Yes" to father's question "Does any of you know whether s/he has mud on her/his forehead?"



$$\blacksquare M'', w_1 \models C_{\{1,2\}}(m_1 \land m_2)$$

 $[!\phi]\psi$: "After ϕ has been truthfully announced, ψ is the case."

Semantics

 $M, w \models [! \varphi] \psi$ iff. $M, w \not\models \varphi$, or else $M_{\varphi}, w \models \psi$

■ M_{φ} is the relativation of M to the worlds where φ holds. The model $M_{\varphi} = (S', R', V')$ is given as follows:

$$S' = \{ w \in S : M, w \models \varphi \}$$
(1)

 $R' = R|_{S' \times S'} \tag{2}$

$$V'(p) = V(p) \cap S' \tag{3}$$

Muddy Children Puzzle: PAL: Initial

(reflexive edges omitted)



To Show: $M, w_1 \models [!\varphi_1][!\varphi_2 \land \varphi_3]K_1m_1 \land K_2m_2$

Nebel, Lindner, Engesser - MAS



JRG

Muddy Children Puzzle: PAL: After Announcement

(reflexive edges omitted)

 $M, w_1 \models [!\varphi_1][!\varphi_2 \land \varphi_3]K_1m_1 \land K_2m_2$ iff. $M, w_1 \not\models \varphi_1$ or else $M_{\varphi_1}, w_1 \models [!\varphi_2 \land \varphi_3]K_1m_1 \land K_2m_2$



$$M_{\varphi_1}, w_1 \models C_{\{1,2\}}(m_1 \lor m_2)$$

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Muddy Children Puzzle: PAL: After Question (reflexive edges omitted)

 $M_{\varphi_1}, w_1 \models [!\varphi_2 \land \varphi_3] K_1 m_1 \land K_2 m_2$

iff. $M_{\varphi_1}, w_1 \not\models \varphi_2 \land \varphi_3$ or else $M_{\varphi_1_{\varphi_2 \land \varphi_3}} K_1 m_1 \land K_2 m_2$



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- Interestingly, $[!\phi]\phi$ is not valid in general.
- Indeed, [!(p ∧ ¬Kp)]¬(p ∧ ¬Kp) is valid. This is related to Moore's paradox saying one cannot know sentences of the form "φ is true and I don't know φ."
 - Let *M* be a model and *w* a world in it.
 - Assume $M, w \models p \land \neg Kp$.
 - Let *N* be the relativation of *M*, $M_{p \land \neg Kp}$.
 - Because $N, w \models p \land \neg Kp$, there must be a successor of w, w', such that $N, w' \neg p$. But as w' is in N, it must also be the case that $N, w' \models p \land \neg Kp$.
 - Contradiction!

Theorem (cf., [1])

For every formula φ with public announcement operator there is a equivalent formula $t(\varphi)$ without public announcement operator.

$$\bullet t(\neg \varphi) = \neg t(\varphi)$$

$$\blacksquare t(\phi \land \psi) = t(\phi) \land t(\psi)$$

$$\blacksquare t(K_i \varphi) = K_i t(\varphi)$$

$$t([!\phi]p) = t(\phi) \to p$$

$$t([!\phi)\neg\psi) = t(\phi \rightarrow \neg [!\phi]\psi)$$

$$= t([!\varphi](\psi \land \chi)) = t([!\varphi]\psi \land [!\varphi]\chi)$$

$$t([!\phi]K_i\psi) = t(\phi \to K_i[!\phi]\psi)$$

$$t([!\varphi][!\psi]\chi) = t([!(\varphi \land [!\varphi]\psi)]\chi)$$

 \Rightarrow PAL does not introduce something really new. But it makes modeling public announcements easier.





- Public communication and observations change what is common knowledge among agents ⇒This kind of dynamics can be modeled using the Public Announcement Operator.
- Public Announcement Logic can be translated to Epistemic Logic.
- The approach can be generalized to updating epistemic models due to arbitary actions (not only announcements).
 Planning based on Dynamic Epistemic Logic is a research area in our group.

Literature



- L. S. Moss, Dynamic Epistemic Logic, Chapter 6, In H. van Dithmarschen, J. Y. Halpern, W. van der Hoek, B. Kooi (eds.) Handbook of Epistemic Logic, College Publications, 2015.
- Y. Shoham, K. Layton-Brown, Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations, Cambridge University Press, 2009.