Multi-Agent Systems

Propositional Logic

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  - Satisfying interpretation = model
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  - Rules how to combine interpretation of single symbols
  - Satisfying interpretation = model
  - Semantics often entails concept of logical implication / entailment
- Specify a calculus that allows to derive new formulae from old ones – according to the entailment relation

Motivation: Deductive Agent

function *action* **in** $(\Delta \in D)$ **out** $(\alpha \in Ac)$
for all $\alpha \in Ac$ **do**
    if $\Delta \vdash_\rho Do(\alpha)$ **then**
        return α
    end if
end for
for all $\alpha \in Ac$ **do**
    if $\Delta \not\vdash_\rho \neg Do(\alpha)$ **then**
        return α
    end if
end for
return *null*

- Δ: Set of formulae written in some logic.
- $\vdash$: Relation that holds between Δs and formulae that can be derived from Δ.

Propositional Logic

Propositional logic: main ideas

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  - representing propositions which cannot be decomposed
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- Logical symbols: propositional connectives such as: and ($\wedge$), or ($\vee$), and not ($\neg$)
- Formulae: built out of atoms and connectives
- Universe of discourse: truth values

Syntax

Syntax

Countable alphabet Σ of propositional variables: $a, b, c, \ldots$
Propositional formulae are built according to the following rule:

φ	$::=$	a	atomic formula
	$\mid$	$\bot$	falsity
	$\mid$	$\top$	truth
	$\mid$	$\neg\varphi'$	negation
	$\mid$	$(\varphi' \land \varphi'')$	conjunction
	$\mid$	$(\varphi' \lor \varphi'')$	disjunction
	$\mid$	$(\varphi' \rightarrow \varphi'')$	implication
	$\mid$	$(\varphi' \leftrightarrow \varphi'')$	equivalence

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& \mid & (\varphi' \leftrightarrow \varphi'') & \text{equivalence}
\end{array}
$$

Parentheses can be omitted if no ambiguity arises.

Operator precedence: $\neg > \land > \lor > \rightarrow = \leftrightarrow$.

Language and meta-language

- $(a \vee b)$ is an expression of the language of propositional logic.
- $\varphi ::= a | \ldots | (\varphi' \leftrightarrow \varphi'')$ is a statement about how expressions in the language of propositional logic can be formed. It is stated using meta-language.
- In order to describe how expressions (in this case formulae) can be formed, we use meta-language.
- When we describe how to interpret formulae, we use meta-language expressions.

Semantics

Semantics: idea

- Atomic propositions can be true $(1, T)$ or false $(0, F)$.
- Provided the truth values of the atoms have been fixed (truth assignment or interpretation), the truth value of a formula can be computed from the truth values of the atoms and the connectives.
- Example:

$$(a \vee b) \wedge c$$

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Logical implication can then be defined as follows:

- φ is implied by a set of formulae Θ iff φ is true for all truth assignments (world states) that make all formulae in Θ true.

Formal semantics

An interpretation (or truth assignment) over Σ is a function:

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A formula ψ is true under $\mathcal{I}$ or is satisfied by $\mathcal{I}$ (symb. $\mathcal{I} \models \psi$):

$$
\begin{array}{rcl}
\mathcal{I} \models a & \text{iff} & \mathcal{I}(a) = T \\
 & \mathcal{I} \models \top & \\
 & \mathcal{I} \not\models \bot & \\
\mathcal{I} \models \neg\varphi & \text{iff} & \mathcal{I} \not\models \varphi \\
\mathcal{I} \models \varphi \wedge \varphi' & \text{iff} & \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \varphi' \\
\mathcal{I} \models \varphi \vee \varphi' & \text{iff} & \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \varphi' \\
\mathcal{I} \models \varphi \to \varphi' & \text{iff} & \text{if } \mathcal{I} \models \varphi \text{ then } \mathcal{I} \models \varphi' \\
\mathcal{I} \models \varphi \leftrightarrow \varphi' & \text{iff} & \mathcal{I} \models \varphi \text{ if and only if } \mathcal{I} \models \varphi'
\end{array}
$$

Example

Given

$$\mathcal{I}: a \mapsto T,\ b \mapsto F,\ c \mapsto F,\ d \mapsto T,$$

Is $((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$ true or false?

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A formula φ is

- satisfiable if there is an $\mathcal{I}$ such that $\mathcal{I} \models \varphi$;
- unsatisfiable, otherwise; and
- valid if $\mathcal{I} \models \varphi$ for each $\mathcal{I}$ (or tautology);
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Formulae φ and ψ are logically equivalent (symb. $\varphi \equiv \psi$) if for all interpretations $\mathcal{I}$,

$$\mathcal{I} \models \varphi \text{ iff } \mathcal{I} \models \psi.$$

Examples

Satisfiable, unsatisfiable, falsifiable, valid?

$(a \lor b \lor \neg c) \land (\neg a \lor \neg b \lor d) \land (\neg a \lor b \lor \neg d)$

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Equivalence? $\neg(a \vee b) \equiv \neg a \wedge \neg b$

- $\leadsto$ Of course, equivalent (de Morgan).

Some obvious consequences

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Theorem

If $\varphi \equiv \psi$, and χ' results from substituting φ by ψ in χ, then $\chi' \equiv \chi$.

Some equivalences

simplifications	$\varphi \rightarrow \psi$	$\equiv$	$\neg\varphi \vee \psi$	$\varphi \leftrightarrow \psi$	$\equiv$	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
idempotency	$\varphi \vee \varphi$	$\equiv$	φ	$\varphi \wedge \varphi$	$\equiv$	φ
commutativity	$\varphi \vee \psi$	$\equiv$	$\psi \vee \varphi$	$\varphi \wedge \psi$	$\equiv$	$\psi \wedge \varphi$
associativity	$(\varphi \vee \psi) \vee \chi$	$\equiv$	$\varphi \vee (\psi \vee \chi)$	$(\varphi \wedge \psi) \wedge \chi$	$\equiv$	$\varphi \wedge (\psi \wedge \chi)$
absorption	$\varphi \vee (\varphi \wedge \psi)$	$\equiv$	φ	$\varphi \wedge (\varphi \vee \psi)$	$\equiv$	φ
distributivity	$\varphi \wedge (\psi \vee \chi)$	$\equiv$	$(\varphi \wedge \psi) \vee (\varphi \wedge \chi)$	$\varphi \vee (\psi \wedge \chi)$	$\equiv$	$(\varphi \vee \psi) \wedge (\varphi \vee \chi)$
double negation	$\neg\neg\varphi$	$\equiv$	φ			
constants	$\neg\top$	$\equiv$	$\bot$	$\neg\bot$	$\equiv$	$\top$
De Morgan	$\neg(\varphi \vee \psi)$	$\equiv$	$\neg\varphi \wedge \neg\psi$	$\neg(\varphi \wedge \psi)$	$\equiv$	$\neg\varphi \vee \neg\psi$
truth	$\varphi \vee \top$	$\equiv$	$\top$	$\varphi \wedge \top$	$\equiv$	φ
falsity	$\varphi \vee \bot$	$\equiv$	φ	$\varphi \wedge \bot$	$\equiv$	$\bot$
taut./contrad.	$\varphi \vee \neg\varphi$	$\equiv$	$\top$	$\varphi \wedge \neg\varphi$	$\equiv$	$\bot$

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  - There are $2^{(2^n)}$ different sets of interpretations.
  - There are $2^{(2^n)}$ (logical) equivalence classes of formulae.

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- Extension of the relation $\models$ to sets Θ of formulae:

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- Contradiction: $\Theta \cup \{\varphi\}$ is unsatisfiable iff $\Theta \models \neg\varphi$

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- One particular calculus: tableaux.

Propositional Tableaux

- Goal: Prove the unsatisfiability of a formula.
- Tableaux algorithm for propositional logic is sound and complete.
- General principle: Break each formula into its components up to the simplest one, where contradiction is easy to spot.

Propositional Tableaux

- A tableaux is a tree. Each branch of that tree corresponds to one attempt to find a model for the input formula.
- Initial Tableaux consists of the node: $\bigwedge\Theta \wedge \neg\varphi$
  - $\Theta \models \varphi$ iff $\bigwedge\Theta \rightarrow \varphi$ is valid iff $\neg(\bigwedge\Theta \rightarrow \varphi)$ is unsatisfiable iff $\bigwedge\Theta \wedge \neg\varphi$ is unsatisfiable
- The tableaux can be incrementally extended by applying rules:
  - And-Rule: If $\varphi \wedge \psi$ is in a branch, then add φ and ψ to it.
  - Or-Rule: If $\varphi \vee \psi$ is in a branch, then add φ to it, add a new branch, and add ψ to it.
  - Implication: If $\varphi \rightarrow \psi$ is in a branch, then add $\neg\varphi$ to it, add a new branch, and add ψ to it.

Propositional Tableaux

- NotNot: If $\neg\neg\varphi$ is in a branch, then add φ to it.
- NotAnd: If $\neg(\varphi \wedge \psi)$ is in a branch, then add $\neg\varphi$ to it, add a new branch, and add $\neg\psi$ to it.
- NotOr: If $\neg(\varphi \vee \psi)$ is in a branch, then add $\neg\varphi$ and $\neg\psi$ to it.
- NotImplication: If $\neg(\varphi \rightarrow \psi)$ is in a branch, then add φ and $\neg\psi$ to that branch.

Propositional Tableaux: Closed Tableaux

- A branch is saturated if no more rule can be applied.
- A branch is closed if it contains formulae φ and $\neg\varphi$.
- A tableaux is closed if all branches are closed.
- If the tableaux is closed, this means no model for the input formula could be found, hence, its negation is valid.