

Social Robotics

Albert-Ludwigs-Universität Freiburg



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Non-parametric Tests



- Wilcoxon signed-rank test
- Wilcoxon rank-sum test (aka Mann-Whitney test)
- Kruskal-Wallis test

- Ranks are natural numbers starting with 1, which get assigned to scores sorted in increasing order.
- Ranks can be assigned to any data which is at least ordinal.
- Ranks are robust against outliers (because ranks are used instead of the actual data).

Example

- Data: 0, 7, 3; Rank: 1, 3, 2
- Data: -100, 99, 98; Rank: 1, 3, 2
- Data: d, a, b; Rank: 3, 1, 2

- In case of ties, the average rank is assigned to the whole group of scores that constitutes the tie.

Example

- Data: 1, 6, 4, 4, 2, 2, 2
- Rank: 1, 7, 5.5, 5.5, 3, 3, 3

Wilcoxon Signed-Rank Test: Motivation



- Likert scales are a popular means of measurement.
- Likert scales in most cases have no interval-scale reading.

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale "never ever!" (1), "maybe" (2), "yes, sure!" (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
- Difference: -1, -1, -1, 0, +2
- Differences without 0: -1, -1, -1, +2
- Ranks: 2, 2, 2, 4

Wilcoxon Signed-Rank Test: Example Continued



Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale "never ever!" (1), "maybe" (2), "yes, sure!" (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
 - Difference: -1, -1, -1, 0, +2; Without 0: -1, -1, -1, +2
 - Ranks: 2, 2, 2, 4
-
- Let $V = \sum_i^n Z_i R_i$ be the sum of the positive ranks ($Z_i = 1$ if difference i is positive, and $Z_i = 0$ else).
 - In the example $V = 4$. Well, so what?

- Imagine two paired samples and consider their rank differences.
- Consider $V = \sum_i^n Z_i R_i$. What could happen?
 - 1 Case $V = 0$: All the rank differences are negative.
 - 2 Case $V = \sum_i^n R_i = \frac{n(n+1)}{2}$: All rank differences are positive.
 - 3 Else: V ranges between 0 and $\frac{n(n+1)}{2}$.
- If the groups do not differ (H_0), then 50% of the differences should be below 0 and 50% above. This is like saying that the median of the difference is 0. And in that case, V should be close to $\frac{\frac{n(n+1)}{2}}{2} = \frac{n(n+1)}{4}$.
- Hence, we will test $H_0 : Mdn = 0$ against its alternatives, and we will do that by using V .

- The nice thing about V is that (for $n > 25$) its distribution is well approximated by a normal distribution $\mathcal{N}(\mu_V, \sigma_V)$ with
 - $\mu_V = \frac{n(n+1)}{4}$
 - $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24}}$

- The nice thing about V is that (for $n > 25$) its distribution under H_0 is well approximated by a normal distribution $\mathcal{N}(\mu_V, \sigma_V)$ with
 - $\mu_V = \frac{n(n+1)}{4}$
 - $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24}}$
- **Proof (Mean):** We already came to this conclusion earlier on Slide 8.

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■ Proof (Variance)

- First, we define $V' = \sum_i^n V'_i$ with

$$V'_i = \begin{cases} 0 & \text{with probability } 0.5 \\ i & \text{with probability } 0.5 \end{cases}$$

- (V' has the same distribution as V , because, for every rank, it either belongs to the sum of V or not with probability 0.5.)
- $\text{Var}(V) = \text{Var}(V') = \sum_i^n \text{Var}(V'_i)$ (independence of V'_i).
- $\text{Var}(V'_i) = E(V_i'^2) - E(V_i')^2 = (0^2 \frac{1}{2} + i^2 \frac{1}{2}) - (\frac{1}{2}i)^2 = \frac{i^2}{4}$
- $\text{Var}(V) = \sum_i^n \text{Var}(V_i) = \sum_i^n \frac{i^2}{4} = \frac{n(n+1)(2n+1)}{24}$.

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale "never ever!" (1), "maybe" (2), "yes, sure!" (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
 - Difference: -1, -1, -1, 0, +2; Without 0: -1, -1, -1, +2
 - Ranks: 2, 2, 2, 4
-
- $V = 4, \mu_V = 4(4 + 1)/4 = 5, \sigma_V = \sqrt{4(4 + 1)(2 \times 4 + 1)/24}$
 - $z = \frac{V - \mu_V}{\sigma_V} = (4 - 5)/2.74 = -0.365$
 - $p = P(z \leq -0.365) + 1 - P(z \leq 0.365) = 0.715$

Example: t-Test

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale "never ever!" (1), "maybe" (2), "yes, sure!" (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1

- Difference: -1, -1, -1, 0, +2

- $\bar{D} = 0.20$, $s_D = 1.30$, $n = 5$

- $t = \sqrt{5} \times 0.20 / 1.30 = 0.344$, $df = 4$

- $p = 0.748$

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots after they have watched the Sci-Fi movie M1, and five participants rate their belief after watching Sci-Fi movie M2. As a measurement instrument, a 3-Point Likert-Scale "never ever!" (1), "maybe" (2), "yes, sure!" (3) was used.

- M1: 1, 1, 2, 2, 2; M2: 2, 3, 3, 3, 2

- H_0 : The two groups are equal.
- Reject H_0 or not?

Wilcoxon Rank-Sum Test: General Setting



- First, all scores are ranked together.
- First group's rank sum: $R_1 = \sum_{i=1}^{n_1} r_{1,i}$
- Second group's rank sum: $R_2 = \sum_{i=1}^{n_2} r_{2,i}$
- First group's W : $W_1 = R_1 - \sum_{i=1}^{n_1} i = R_1 - \frac{n_1(n_1+1)}{2}$
- Second group's W : $W_2 = R_2 - \sum_{i=1}^{n_2} i = R_2 - \frac{n_2(n_2+1)}{2}$
- $W_1 + W_2 = R_1 - \frac{n_1(n_1+1)}{2} + R_2 - \frac{n_2(n_2+1)}{2} = n_1 n_2$
- **Note:** The Wilcoxon Rank-Sum Test is also known as Mann-Whitney U-Test, and W is also called U . There are various ways of defining W (resp. U), which are all equal! R uses the statistics W the way shown above.

- For larger samples ($n_1 > 10, n_2 > 10$), $W \sim \mathcal{N}(\mu_W, \sigma_W)$:
 - $\mu_W = \frac{n_1 n_2}{2}$
 - $\sigma_W = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}}$
 - Also see simulation in `lecture11.Rmd` in the git repository slides.
- Again, we can calculate z-values to decide whether or not W is extreme, i.e., whether or not to reject H_0 .

Wilcoxon Rank-Sum Test: Example

Continued



Example

■ M1: 1, 1, 2, 2, 2

■ M2: 2, 3, 3, 3, 2

■ All Scores: 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 2

■ Ranks: 1.5, 1.5, 5, 5, 5, 5, 9, 9, 9, 9, 5

■ $R_1 = 18$, $W = 18 - 15 = 3$

■ $z = \frac{\frac{3 - (5 \times 5)}{2}}{\sqrt{\frac{5 \times 5(5 + 5 + 1)}{12}}} = -2.298$

■ $p = P(z \leq -2.298) + 1 - P(z \leq 2.298) = 0.022$

Example

- M1: 1, 1, 2, 2, 2
- M2: 2, 3, 3, 3, 2
- $\bar{X}_1 = 1.6, \bar{X}_2 = 2.6, s_1^2 = 0.3, s_2^2 = 0.3, n = 5, df = 8$
- $t = \sqrt{n} \times \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{s_1^2 + s_2^2}} = \sqrt{5} \times \frac{1.6 - 2.6}{\sqrt{0.3 + 0.3}} = -2.887$
- $p = P(t \leq -2.887) + 1 - P(t \leq 2.887) = 0.020$
- For a simulation comparing Wilcoxon and t-Test see `lecture11.Rmd` in the git repository.

- Also for rank-based methods, there is an analog to ANOVA that can cope with more than two groups: **Kruskal-Wallis Test**. As for ANOVA, H_0 reads “There is no difference between the groups”.
- First, the scores of all groups are ranked together (like for Wilcoxon Rank-Sum Test).
- The test statistics is called H :
 - $H = (N - 1) \frac{\sum_i^p n_i (\bar{r}_i - \bar{r})^2}{\sum_i^p \sum_j^{n_i} (r_{ij} - \bar{r})^2}$, with $N = \sum_i^p n_i$, $\bar{r}_i = \frac{\sum_j^{n_i} r_{ij}}{n_i}$, $\bar{r} = \frac{N+1}{2}$
 - H can be simplified to $H = \frac{12}{N(N+1)} \sum_i^p n_i \bar{r}_i^2 - 3(N+1)$
- $H \sim \chi_{p-1}^2$, with p being the number of groups.

Example

- M1: 1, 1, 2, 2, 2; Ranks: 2.5, 2.5, 8.5, 8.5, 8.5
 - M2: 2, 3, 3, 3, 2; Ranks: 8.5, 14, 14, 14, 8.5
 - M3: 1, 2, 2, 1, 2; Ranks: 2.5, 8.5, 8.5, 2.5, 8.5
-
- $\bar{r}_1 = 6.1, \bar{r}_2 = 11.8, \bar{r}_3 = 6.1, N = 15, \bar{r} = (15 + 1)/2 = 8$
 - $H = \frac{12}{15 \times 16} \times 5(37.21 + 139.24 + 37.21) - 3 \times 16 = 5.41$
 - $p = 1 - P(\chi^2 \leq 5.41) = 0.067$
 - R will report different values, see next slide to learn why.

If there are long ties (i.e., a lot of scores getting the same rank), the variance of the statistics become smaller and thus some corrections have to be applied.

- The V-statistics's standard deviation becomes:

- $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24} - \sum_i^k \frac{t_i^3 - t_i}{48}}$ (cf., slide 9)

- The W-statistics's standard deviation becomes:

- $\sigma_W = \sqrt{\frac{n_1 n_2}{12} \left((n_1 + n_2 + 1) - \sum_i^k \frac{t_i^3 - t_i}{(n_1 + n_2)(n_1 + n_2 - 1)} \right)}$ (cf., slide 16)

- And the H-statistics can be corrected by dividing H by the term $corr = 1 - \frac{\sum_i^k (t_i^3 - t_i)}{N^3 - N}$

- In the example: $corr = 1 - \frac{(4^3 - 4) + (8^3 - 8) + (3^3 - 3)}{(15^3 - 15)}$

- The corrected H value then is $H_{corr} = 6.56$

- Because all this is rather tedious, you are allowed to skip these corrections in your assignments (also in the exam).

- Categorical Scale
 - χ^2 -statistics (χ^2 -distributed)
- Interval Scale
 - Variance known: z-statistics (normally distributed)
 - Variance unknown (but equal): t-statistics (Student's t distribution), F-statistics (F-distributed)
- Ordinal Scale
 - W-, V-statistics (both normally distributed), H-statistics (χ^2 -distributed)

- We started out defining four types of hypotheses
 - 1 Directional difference hypotheses
 - 2 Undirectional difference hypotheses
 - 3 Directional relationship hypotheses
 - 4 Undirectional relationship hypotheses
- We can so far only deal with (1) and (2). This is going to be fixed during the next statistics block starting from January 16th. The timeline:
 - **Next:** We will have a mock exam on January 9th.
 - **NextNext:** We will have another Reading Session on January 11.
 - **NextNextNext:** Correlation and Regression for testing relationship hypotheses from January 16th.
 - **New Exam Date: February 21th!**

Merry Christmas!



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Sketches

Intentionally left blank :-)



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