Dynamic Epistemic Logic

Chapter 4 - Action models

4.4 Semantics of action model logic

(cont.)
Proof: Let (M,t) be arbitrary. Show that $M, t \models [(m, s); (m', s')]\phi$ iff $M, t \models [m, s][m', s']\phi$.
Sufficient to show: $M \bigotimes (m; m')$ is isomorphic to $(M \bigotimes m) \bigotimes m')$.
Isomoporphic domains: Let $(t, (s, s')) \in dom(M \bigotimes (m, m'))$. Then:
$M, t \models pre''((s, s')) = \langle m, s\rangle pre'(s')$
This holds iff $M, t \models pre(s) \wedge [m, s]pre'(s')$
This holds iff $M, t \models pre(s)$ (1)
and $M, t \models [m, s]pre'(s')$ (2)
From (1) $(t, s) \in dom(M \bigotimes m)$ (3).
From (2) and (3) $(M \bigotimes m, (t, s)) \models pre'(s')$.
This implies $((t, s), s') \in dom(M \bigotimes m) \bigotimes m')$. Conversely, we also get $(t, (s, s')) \in dom(M \bigotimes (m, m'))\forall((t, s), s') \in dom((M \bigotimes m) \bigotimes m'))$
Accessibility relations: Assume that $(t, (s, s')) \sim_a (t_1, (s_1, s'_1))$. This holds iff:
$t \sim_a t_1$ and $(s, s') \sim_a (s_1, s'_1)$ iff
$t \sim_a t_1$ and $s \sim_a s'$ and $s' \sim_a s'_1$ iff
$(t, s) \sim_a (t_1, s_1)$ and $(s' \sim_1 s'_1)$ iff
$((t, s), s') \sim_a ((t_1, s_1), s'_1)$.
Valuations: clear.

Propositions: Let $\alpha, \beta, \gamma \in \mathcal{L}^{act}_{KC\bigotimes}(A, P)$. Then $((\alpha \cup \beta); \gamma)$ is equivalent to $(\alpha; \gamma) \cup (\beta; \gamma)$ and $(\alpha; (\beta \cup \gamma))$ is equivalent to $(\alpha; \beta) \cup (\alpha; \gamma)$.

Proposition: Let $\alpha, \beta \in \mathcal{L}^{act}_{KC\bigotimes}(A, P)$ and $\phi \in \mathcal{L}^{stat}_{KC\bigotimes}(A, P)$. Then $[\alpha \cup \beta]\phi$ is equivalent to $[\alpha]\phi \wedge [\beta]\phi$.

4.5 Bisimilarity and Action Emulation

Can two action models be bisimilar? $\rightarrow$ Yes.
Does the application of bisimilar action models to bisimilar epistemic states, lead to bisimilar successor states? $\rightarrow$ Yes.

Do we even need bisimilarity of actions models for that? $\rightarrow$ No. Weaker notion of emulation is enough.

Example 1 *m_1 and m_2 are not bisimilar but always behave in the same way (similar enough)*

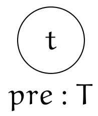


Figure 1: m_1

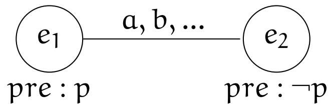
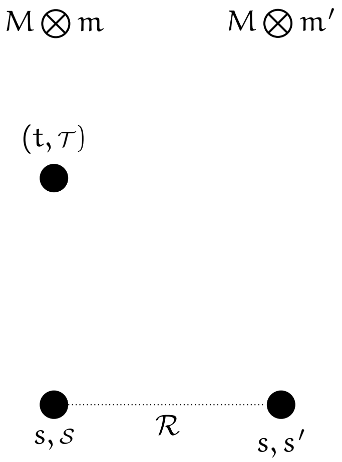


Figure 2: m_2

Proposition(preservation of bisimilarity): Let (M,s) and (M',s') be two epistemic states that are bisimilar. Let (m,$\mathcal{S}$) with $m = \langle S, \sim, pre \rangle$ be applicable in (M,s). Then $(M \bigotimes m, (s, \mathcal{S})) \; \underline{\leftrightarrow} \; (M' \bigotimes m, (s', \mathcal{S}))$.
Proof: $(m, \mathcal{S})$ is applicable in (M',s'), since $M, s \models pre(\mathcal{S})$ and $(M, s) \; \underline{\leftrightarrow} \; (M', s')$ implies $(M', s') \models pre(\mathcal{S})$. Let $\mathcal{R} : (M, s) \; \underline{\leftrightarrow} \; (M', s')$. Then the bisimulation between $\mathcal{R}'$: $(M \bigotimes m, (s, \mathcal{S})) \; \underline{\leftrightarrow} \; (M' \bigotimes m, (s', \mathcal{S}))$ between the successor states can be defined as follows: $\mathcal{R}' : ((t, \mathcal{T}), (t', \mathcal{T}'))$ iff $\mathcal{R}(t, t')$ and $\mathcal{T} = \mathcal{T}'$

Proposition: Given an epistemic modelM and action models $\mathcal{E} : m \leftrightarrows m'$. Then $M \bigotimes m$ $\underline{\leftrightarrow}$ m'.
Proof: Let $m = \langle S, \sim, pre \rangle$ and $m' = \langle S', \sim', pre' \rangle$. Define $\mathcal{R} : M \bigotimes m \; \underline{\leftrightarrow} \; M' \bigotimes m'$ as $\mathcal{R}((s, \mathcal{S}), (s', \mathcal{S}'))$ iff s=s' and $\mathcal{E}(s, s')$. Show that $\mathcal{R}$ is a total bisimulation between $M \bigotimes m$ and $m \bigotimes m'$.

Forth:



Let $(s, \mathcal{S}) \sim_a (t, \mathcal{T})$ and $\mathcal{R}((s, \mathcal{S}), (s, s'))$. Then $s \sim_a t$ and $\mathcal{E}(s, s')$.

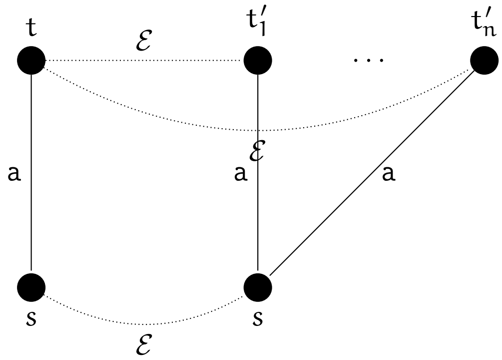


Figure 3: m

Then there are $t'_1, ..., t'_n$ s.t. $\mathcal{E}(t, t'_1), ..., \mathcal{E}(t, t'_n)$ s.t. $s' \sim_a t'_1, ..., s' \sim_a t'n$ and $pre(t) \models pre'(t'_1) \vee ... \vee pre'(t'_n)$.
We know that $(t, \mathcal{T}) \in dom(M \bigotimes m)$. So, $M, t \models pre(t) \rightarrow M, t \models pre'(t'_1), ..., pre'(t'_n)$. So, there is an $i \in (1, ..., n)$ s.t. $M, t \models pre'(t'_i)$. Therefore $(t, t'_1) \in dom(M \bigotimes m')$. Furthermore, $\mathcal{R}((t, \mathcal{T}), (t, t'_1))$. By definition of $\mathcal{R}$ and $(s, s') \sim_a (t, t'_i)$ since $s \sim_a t$ and $s' \sim'_a t'_i$.
Back:Similar
Valuations: $\mathcal{R}((s, \mathcal{S}), (s', \mathcal{S}'))$ implies s=s'. There are no ontic effects therefore the valuation does not change after action application

Remark: For action models with propositional preconditions action emulation fully characterizes the effect of action application. Also if $M \bigotimes m) \underline{\leftrightarrow} (M \bigotimes m')$ then $m \leftrightarrows$ m'

Example 2

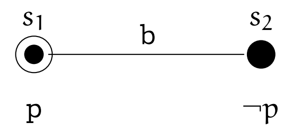


Figure 4: Model m

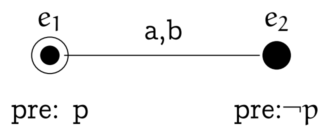


Figure 5: Action a

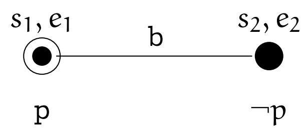


Figure 6: $m \bigotimes a$

4.6 Validities and axiomatisation

Validities in PA:
$\langle\alpha\rangle\phi \rightarrow [\alpha]\phi$ is not valid. (Reason is nondeterminism).
But $[\alpha \vee \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$ is valid.
Consider single pointed models only, get rid of nondeterminism.

Proposition: (atomic permanence). $[m, s]p \leftrightarrow (pre(s) \rightarrow p)$ is valid.
Proposition: $[m, s]p \leftrightarrow (pre(s) \rightarrow p)$ is valid.
Proposition: $[m, s]\neg\phi \leftrightarrow (pre(s) \rightarrow \neg[m, s]\phi)$ is valid.
Proposition: $[m, s](\phi \wedge \psi) \leftrightarrow ([m, s]\phi \wedge [m, s]\psi)$ is valid.
What about PA principle $[\phi]K_a\psi \leftrightarrow (\phi \rightarrow K_a[\phi]\psi)$?
It does not directly generalise to action model logic.

Example 3

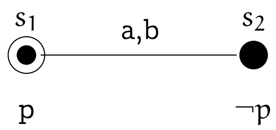


Figure 7: Before, s_1

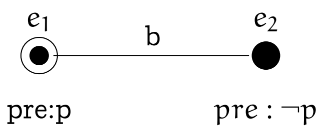


Figure 8: Read_a, e_1

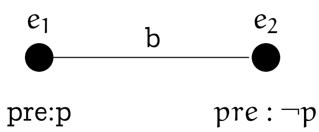


Figure 9: Read_a, e_2

On the one hand:
$\mathrm{Before}, s_1 \models p \to K_b[\mathrm{Read}_a, e_1]p$, since
$\mathrm{Before}, s_1 \models [\mathrm{Read}_a, e_1]p$ and
$\mathrm{Before}, s_2 \models [\mathrm{Read}_a, e_1]p$, since
$(\mathrm{Before}, s_1 \models pre(e_1) \to (\mathrm{Before} \bigotimes \mathrm{Read}, (s_1, e_1)) \models p)$ and $\mathrm{Before}, s_2 \not\models pre(e_1)$.
On the other hand:
$\mathrm{Before}, s_1 \not\models [\mathrm{Read}_a, e_1]K_b p$, since
$\mathrm{Before}, s_1 \models pre(e_1)$, but $(\mathrm{Before} \bigotimes \mathrm{Read}_a, (s_1, e_1)) \not\models K_b p$.
<u>Intuition:</u> Agent b may mistake action (Read_a, e_1) for action (Read_a, e_2) when observing it. Hence when observing (Read_a, e_1) he does not learn that p is true.
$\mathrm{Before}, s_1 \not\models [\mathrm{Read}, e]K_b p \leftrightarrow (pre(e_1) \to K_b[\mathrm{Read}_a, e_1]p)$. Hence $[m, s]K_a\phi \leftrightarrow (pre(s) \to K_a[m, s]\phi)$ is not valid!

<u>Proposition:</u> $[m, s]K_a\phi \leftrightarrow (pre(s) \to \bigwedge_{s\sim_a t} K_a[m, t]\phi)$ js now valid.
<u>Proof:</u> Prove the dual. $(pre(s) \wedge \bigwedge_{s\sim_a t} \hat{K}_a\langle m, t\rangle\phi) \leftrightarrow \langle m, s\rangle\hat{K}_a\phi$ is valid.
Let $M = \langle S, \sim, V\rangle$ and $m = \langle S', \sim, pre\rangle$.
<u>"$\Rightarrow$":</u> Assume that $M, s \models pre(s)$ and there is a $\mathcal{T} \in S'$ with $s \sim_a t$ and $M, t \models \hat{K}_a\langle m, t\rangle\phi$.
Hence $(s, \mathcal{S}) \in dom(M \bigotimes m)$ and there is a state $t \in S$ with $s \sim_a t$ and $M, t \models \langle m, t\rangle\phi$.
Thus $M, t \models pre(\mathcal{T})$ and $(t, \mathcal{T}) \in dom(M \bigotimes m)$ and $(M \bigotimes m, (t, \mathcal{T})) \models \phi$.

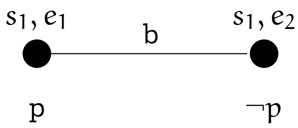


Figure 10: $(Before \bigotimes Read_a, (s_1, e_1))$

WIth $s \sim_a t$ and $\mathcal{S} \sim_a \mathcal{T}$, we get $(s, \mathcal{S}) \sim_a (t, \mathcal{T})$. Hence $(M \bigotimes m), (s, \mathcal{S})) \models \hat{K}_a\phi$.
So, $M, s \models \langle m, s\rangle \hat{K}_a\phi$.
"$\Leftarrow$": Assume that $M, s \models \langle m, s\rangle\hat{K}_a\phi$. So $M, s \models pre(\mathcal{S})$ and $(M \bigotimes m, (s, \mathcal{S})) \models \hat{K}_a\phi$.
So there is a $(t, \mathcal{T}) \in S \times S'$ s.t. $(s, \mathcal{S}) \sim_a (t, \mathcal{T})$ and $(M \bigotimes m, (t, \mathcal{T}))\phi$
Thus $s \sim_a t$ and $\mathcal{S} \sim_a \mathcal{T}$. Moreover, $M, t \models \langle m, t\rangle\phi$ with $s \sim_a t$, we get $M, s \models \hat{K}_a\langle m, t\rangle\phi$.
With $s \sim_a t$, we get $M, s \models \bigvee_{s \sim_a t} \hat{K}_a\langle m, t\rangle\phi$.

Proposition: (Actions and Common knowledge). Given an action model $(m, \mathcal{S})$ and formulas $\chi_\mathcal{T} \forall \mathcal{T} \sim_B \mathcal{S}$. If for all $a \in B$ and for $\mathcal{U} \sim_a \mathcal{T} \models \chi_\mathcal{T} \rightarrow [m, \mathcal{T}]\phi$ and $\models (\chi_\mathcal{T} \wedge pre(\mathcal{T})) \rightarrow K_a\chi_\mathcal{U}$, then $\models \chi_\mathcal{S} \rightarrow [m, \mathcal{S}]C_B\phi$.
Proof: Let $m = \langle S, \sim, pre\rangle$. To show $\models \chi_\mathcal{S} \rightarrow [m, \mathcal{S}]C_B\phi$.
Assume an arbitrary M,s s.t. in $M, s \models \chi_{smallS}$ and assume that $M, s \models pre(\mathcal{S})$.
To show $(M \bigotimes m, (s, \mathcal{S})) \models C_B\phi$.
Assume an arbitrary state $(u, \mathcal{U}) \in dom(M \bigotimes m)$. We show that $(M \bigotimes m, (u, \mathcal{U}))\phi$ by induction on the path length from $(s, \mathcal{S})$, to $(u, \mathcal{U})$.
$(M \bigotimes m, (u, \mathcal{U})) \models \phi$ and $M, u \models \chi_\mathcal{U}$.
Base case u=s. Follows from $\models \chi_\mathcal{T} \rightarrow [m, t]\phi$ for t=s and M,s.
Inductive case n+1. Then there is a state $(t, \mathcal{T})$, s.t. $(s, \mathcal{S}) \sim_B (t, \mathcal{T}) \sim_a (u, \mathcal{U})$.
With I.H. we get $(M \bigotimes m, (t, \mathcal{T})) \models \phi$ and $M, t \models \chi_\mathcal{T}$. With $M, t \models \chi_\mathcal{T}$ and $M, t \models pre(\mathcal{T})$ and $t \sim_a u$ and assumption $\models (\chi_\mathcal{T} \wedge pre(\mathcal{T})) \rightarrow K_a\chi_\mathcal{U}$ we get $M, t \models K_a\chi_\mathcal{U}$.
With $t \sim_a u$, $M, u \models \chi_\mathcal{U}$. With assumed validity $\models \chi_\mathcal{T} \rightarrow [m, \mathcal{T}]\phi$ we get $M, u \models [m, \mathcal{U}]\phi$.
With $(u, \mathcal{U}) \in dom(M \bigotimes m)$ we get $M, u \models pre(\mathcal{U})$.
Hence $(M \bigotimes m, (u, \mathcal{U})) \models \phi$.

Proof system for AMC for action model logic:
$\mathcal{S}5C$ +
atomic permanence
action and negation
action and conjunction
action and knowledge
action composition
nondeterministic choice.
From ϕ infer $[m, s]\phi$ (neccessitation of [m,s])
Given (m, s) and χ_t for $t \sim_a s$. If for all $a \in B$ and $u \sim_a t$ $\chi_t \rightarrow [m, t]\phi$ and

$(\chi_t \wedge pre(t)) \to K_a\chi_u$, then infer $\chi_s \to [m, s]C_B\phi$.

Example 4 *Derivation in AMC of* $\models [Read_a, e_1]K_a p$

1. $p \to p$ *taut.*

2. $[Read_a, e_1]p \leftrightarrow (p \to p)$ *atomic permanence,* $pre(e_1) = p$

3. $[Read_a, e_1]p$ *1,2,prop. reasoning*

4. $K_a[Read_a, e_1]p$ *e, necc. of* K_a

5. $p \to K_a[Read_a, e_1]p$ *4, prop. reasoning, weakening*

6. $[Read_a, e_1]K_a \leftrightarrow (p \to K_a[Read_a, e_1]p)$ *action and knowledge*

7. $[Read_a, e_1]K_a p$ *5,6, prop. reasoning*