Dynamic Epistemic Logic

Chapter 3 - Public announcements

3.6 Announcements and Common knowledge

$[\phi]K_a\psi \leftrightarrow (\phi \rightarrow K_a[\phi]\psi$
Generalize to common knowledge. $[\phi]C_B\psi \leftrightarrow (\phi \rightarrow C_B[\phi]\psi)$.
<u>This is invalid!</u>

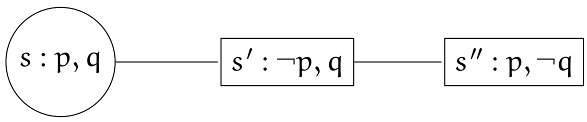

Figure 1: Before the announcement of p

Figure 2: After the announcement of p

$M,s \models [p]C_{ab}q$
$M,s \not\models p \rightarrow C_{ab}q$

<u>Proposition</u> (Announcement and Common knowledge)
If $\chi \rightarrow [\phi]\psi$ and $(\chi \wedge \psi) \rightarrow E_B$ are valid then $\chi \rightarrow [\phi]C_B\psi$ is valid.
<u>Proof</u>
Let M,s be arbitrary and suppose $M,s \models \chi$. We want to show $M,s \models [\phi]C_B\psi$.
Suppose $M,s \models \psi$ and t be in the domain of $M|_\phi$ s.t. $s \sim_B t$. We prove $M|_\psi, t \models \psi$ by induction over the path length from s to t.
If length=0 then s=t and $M|_\psi, s \models \psi$ which follows from $M,s \models \chi$ and the validity $\chi \rightarrow [\phi]\psi$
If length=n+1 with $a \in B$ and $u \in M|_\phi$ s.t. $s \sim_a u \sim_B t$.
From $M,s \models \chi$, $M,s \models \phi$, from the validity of $(\chi \wedge \phi) \rightarrow E_B\chi$ and $s \sim_a u$ follows, $M,s \models \chi$.
Because $u \in M|_\phi$ we know that $M,s\phi$. Now apply the induction hypothesis. So we have $M|_\psi, t \models \psi$

Corollary
$[\phi]\psi$ is valid iff $[\phi]C_B\psi$ is valid.
Proof
"$\leftarrow$" : trivial
"$\rightarrow$: X=T in the last proposition

3.7 Unsuccessful Updates

Definition 1 *Given a fomula $\phi \in \mathcal{L}_{KC[]}$ and an epistemic state M,s:*
ϕ is a successful formula if $[\phi]\phi$ is valid.
ϕ is a unsuccessful formula if it is not successful.
ϕ is a successful update in M,s if M,s $\models \langle\phi\rangle\phi$.
ϕ is unsuccessful in M,s if M,s$\models \langle\phi\rangle\neg\phi$.

Note:
Updates with true successful formulas are always successful.
Updates with unsuccessful formulas can be successful.

Question: Can we characterize successful formulas syntactically?
Not trivial, since it is possible that ϕ and ψ are successful but their conjuction or disjunction are not. (Exercise: find such formulas and discuss!)

Proposition (Public knowledge updates)
Let $\phi \in \mathcal{L}_{KC[]}$. Then $[C_A\phi]C_a\phi$ is valid.
Proof
Let M,s be arbitrary. If $C_a\phi$ is not true in s, then $[C_A\phi]C_a\phi$ is true. If Ms,$\models C_a\phi$, then M,t $\models \phi$ if $s \sim_a t$, and by the fact that M,t$\models C_A\phi$. Clearly $t \in dom(M)$ iff $t \in dom(M|_{C_A\phi})$, which means that $M|_{C_A\phi}, s \models C_A\phi$. Hence M,s$\models [C_A\phi]C_A$

What if $B \subsetneq A$? Is $[C_B\phi]C_B\phi$ still valid? It is not!
Counterexample: Recall the example from earlier that showed that $[p\wedge\neg K_a p](p\wedge\neg K_a p)$ is not valid. Let B={b}. Now consider the update formula $[C_B(p\wedge\neg K_a p)]C_B(p\wedge\neg K_a p)$. This is not valid, obviously.
We call a model M' a submodel of M if $dom(M') \subseteq dom(M)$ and $\sim$ and V are restricted accordingly.

Definition 2 $\mathcal{L}^0_{KC[]}$
$\mathcal{L}^0_{KC[]}(A, P)$ is the following fragment of $\mathcal{L}_{KC[]}$:
$\phi ::= p|\neg p|\phi \wedge \phi|\phi \vee \phi|K_a\phi|C_B\phi|[\neg\phi]\phi$

Proposition(Preservation)
Fragment $\mathcal{L}^0_{KC[]}(A, P)$ is preserved under submodels.
Proof
By structural induction.
Best case: p and $\neg p$ are trivial.
Assume M' is a submodel of M with $s \in dom(M) \cap dom(M')$. M,s $\models p$ iff M,s$\models p$.
$\phi \wedge \psi$ and $\phi \vee \psi$
M,s $\models \phi \wedge \psi$ iff
M,s $\models \phi$ and $M, s \models \psi$ iff (I.H.)
M',s $\models \phi$ and M',s$\models \psi$ iff
M',s $\models \phi \wedge \psi$
$K_a\phi$:
Let $M = \langle S, \sim, V\rangle$ be given and $M' = \langle S', \sim', V'\rangle$ a submodel of M. Let $s \in S'$. Suppose M,s$\models K_a\phi$. Let $s' \in S'$ and $s \sim'_a s'$. Then M,s'$\models \phi$. By I.H.: M',s'$\models \phi$. Therefore M',s$\models K_a\phi$.
$C_B\phi$: Analogous.
$[\neg\phi]\psi$:
Suppose $[\neg\phi]\psi$ and suppose for contradiction. M',s $\not\models [\neg\phi]\psi$. This implies M',s $\models \neg\phi$ and $M'|_{\neg\phi}, s \not\models \psi$.
Using the contrapositive of the I.H. we arrive at M,s$\models \neg\phi$. Moreover $M'|_{\neg\phi}$ is a submodel of $M|_{\neg\phi}$ because $t \in S'$ only survives if M',t$\models \neg\phi$. Again I.H. M,t$\models \neg\phi$, so $[\![\neg\phi]\!]_{M'} \subseteq [\![\neg\phi]\!]_M$. But from M,s$\models [\neg\phi]\psi$ and M,s$\models \neg\phi$ it follows $M|_{\neg\phi}, s \models \psi$, therefore $M'|_{\neg\phi}, s \models \psi$, which is a contradiction.

Corollary:
Let $\phi \in \mathcal{L}^0_{KC[]}(A, P)$ and $\psi \in \mathcal{L}_{KC[]}(A, P)$. Then $\phi \to [\psi]\phi$ is valid because $[\psi]$ creates a submodel.
Corollary:
Let $\phi \in \mathcal{L}^0_{KC[]}(A, P)$ then $\phi \to [\phi]\phi$ is valid.
Corollary:
Let $\phi \in \mathcal{L}^0_{KC[]}(A, P)$ then $[\phi]\phi$ is valid.

Proposition:
Incosistent formulas are successful.

Example 1 $[p \wedge \neg p]p \wedge \neg p$

3.8 Axiomatisation

3.8.1 PA (without Common Knowledge)

Axiom schemata for $\mathcal{L}_{K[]}(A, P)$ with $a \in A$ and $p \in P$:
All instantiations of propositional tautologies [tautologies]
$K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi)$ [distribution of K_a over $\to$]
$K_a\phi \to \phi$ [truth]
$K_a\phi \to K_aK_a\phi$ [positive introspection]
$\neg K_a\phi \to K_a\neg K_a\phi$ [negative introspection]
$[\phi]p \leftrightarrow \phi \to p$ [atomic permanence]
$[\phi]\neg\psi \leftrightarrow (\phi \to \neg[\phi]\psi)$ [announcement plus negation]
$[\phi](\psi \wedge \chi) \leftrightarrow ([\phi]\psi \wedge [\phi]\chi)$ [announcement plus conjunction]
$[\phi]K_a\psi \leftrightarrow (\phi \to K_a[\phi]\chi$ [announcement plus knowledge]
$[\phi][\psi]\chi \leftrightarrow [\phi \wedge [\phi]\psi]\chi$ [announcement plus composition]

Inference rules:
From $\phi \wedge \phi \to \psi$, infer ψ [modus ponens]
From ϕ infer $K_a\phi$ [neccesitation]

Example 2 *We want to show* $\vdash [p]K_ap$
1. $p \to p$ *taut.*
2. $[p]p \leftrightarrow p \to p$ *atomic permanence*
3. $[p]p$ *1,2,prop.*
4. $K_a[p]p$ *3,nec.*
5. $p \to K_a[p]p$ *4,prop.*
6. $[p]K_ap \leftrightarrow (p \to K_a[p]p$ *ann.+know.*
7. $[p]K_ap$ *5,6,prop.*

Theorem 1 *The axiomatisation PA(A,P) is sound and complete*