Dynamic Epistemic Logic

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3.5 Principles of Public Announcement Logics

Today, we will prove some valid formulas of the language $\mathcal{L}_{K[]}$ that will ultimately allow us to reduce $\mathcal{L}_{K[]}$ to $\mathcal{L}_K$ and get rid of announcement modalities.

Proposition (Functionality). *It is valid that* $\langle\phi\rangle\psi \to [\phi]\psi$.

Proof. Let M, s be arbitrary. Assume that $M, s \models \langle\phi\rangle\psi$. This is true if and only if $M, s \models \phi$ and $M|_\phi, s \models \psi$. This implies that $M, s \models \phi$ implies $M|_\phi, s \models \psi$, i. e., $M, s \models [\phi]\psi$. □

This validity raises the question whether the implication in the opposite direction is also valid. However, this is clearly not the case, as the following counterexample shows: Consider a model M with a single state s where atom p is false. Then $M, s \models [p]p$, but $M, s \not\models \langle p\rangle p$.

Proposition (Partiality). $\langle\phi\rangle\top$ *is not valid.*

Proof. In any epistemic state M, s with $M, s \not\models \phi$, we have $M, s \not\models \langle\phi\rangle\top$. □

Proposition (Negation). $[\phi]\neg\psi \leftrightarrow (\phi \to \neg[\phi]\psi)$ *is valid.*

Proof. Omitted. Note that the biimplication can be equivalently written as $[\phi]\neg\psi \leftrightarrow (\neg\phi \vee \langle\phi\rangle\neg\psi)$. □

Proposition. *All of the following are equivalent:*

1. $\phi \to [\phi]\psi$
2. $\phi \to \langle\phi\rangle\psi$
3. $[\phi]\psi$

Proof. We show that (1) and (3) are equivalent. Assume $M, s \models \phi \to [\phi]\psi$. This holds iff $M, s \models \phi$ implies $M, s \models [\phi]\psi$. This in turn holds iff $M, s \models \phi$ implies ($M, s \models \phi$ implies $M|_\phi, s \models \psi$). This is equivalent to ($M, s \models \phi$ and $M, s \models \phi$) implies $M|_\phi, s \models \psi$, which is the same as saying that $M, s \models \phi$ implies $M|_\phi, s \models \psi$. This is the definition of $M, s \models [\phi]\psi$. We should now also show that (1) and (2) are equivalent, or that (2) and (3) are equivalent. This is an easy homework exercise, and hence omitted. □

Proposition. *All of the following are equivalent:*

1. $\langle\phi\rangle\psi$
2. $\phi \wedge \langle\phi\rangle\psi$
3. $\phi \wedge [\phi]\psi$

Proof. Clear. □

Proposition (Composition). $[\phi][\psi]\chi$ *is equivalent to* $[\phi \wedge [\phi]\psi]\chi$.

Proof. For arbitrary M, s, we have $s \in M|_{\phi\wedge[\phi]\psi}$ iff $M, s \models \phi \wedge [\phi]\psi$ iff $M, s \models \phi$ and ($M, s \models \phi$ implies $M|_\phi, s \models \psi$) iff $s \in M|_\phi$ and $M|_\phi, s \models \psi$ iff $s \in M|_\phi|_\psi$. □

Let us now study how knowledge changes with announcements. We find that $[\phi]K_a\psi$ is *not* equivalent to $K_a[\phi]\psi$. Here is a counterexample (recall the Hexa model from above): $\text{Hexa}, 012 \models [1_a]K_c0_a$, but on the other hand, $\text{Hexa}, 012 \not\models K_c[1_a]0_a$.

Proposition (Knowledge). $[\phi]K_a\psi$ *is equivalent to* $\phi \to K_a[\phi]\psi$.

Proof. Assume that $M, s \models \phi \to K_a[\phi]\psi$. This holds iff $M, s \models \phi$ implies $M, s \models K_a[\phi]\psi$ iff $M, s \models \phi$ implies ($M, t \models \phi$ implies $M|_\phi, t \models \psi$) for all t such that $(s, t) \in \sim_a$ iff $M, s \models \phi$ implies ($M, t \models \phi$ and $(s, t) \in \sim_a$ implies $M|_\phi, t \models \psi$) for all $t \in S$ iff $M, s \models \phi$ implies (($s, t) \in \sim_a$ implies $M|_\phi, t \models \psi$) for all $t \in [[\phi]]$ iff $M, s \models \phi$ implies ($M|_\phi, s \models K_a\psi$) iff $M, s \models [\phi]K_a\psi$. □

Proposition (Reduction). *All of the following schemas are valid:*

1. $[\phi]p \leftrightarrow (\phi \to p)$ *for all* $p \in P$
2. $[\phi](\psi \wedge \chi) \leftrightarrow ([\phi]\psi \wedge [\phi]\chi)$
3. $[\phi](\psi \to \chi) \leftrightarrow ([\phi]\psi \to [\phi]\chi)$
4. $[\phi]\neg\psi \leftrightarrow ([\phi] \to \neg[\phi]\psi)$
5. $[\phi]K_a\psi \leftrightarrow (\phi \to K_a[\phi]\psi)$
6. $[\phi][\psi]\chi \leftrightarrow [\phi \wedge [\phi]\psi]\chi$

Proof. We already showed (4), (5), and (6). The others are an easy homework exercise. □

Note: Using this proposition, one can reduce any $\mathcal{L}_{K[]}$ formula to an $\mathcal{L}_K$ formula. This means that both logics are equally expressive, and that we can use $\mathcal{L}_K$ theorem provers or model checkers for $\mathcal{L}_{K[]}$ as well.