Dynamic Epistemic Logic

Chapter 2 - Multi agent S5

Axiomatisation and Common knowledge

2.3 Axiomatisation

1- Semantic derivation of valid formulas via Kripke models.
2- Syntatic derivation of valid formulas via axioms.
Modal logic **K**:
- all instantiations of propositional tautologies (Prop)
- $K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi)$ (K)
- From $\phi \wedge \phi \to \psi$, we can infer ψ (MP, modus ponens)
- From ϕ, we can infer $K_a\phi$ (Nec, neccesitation)

Definition 1 *Derivation*
Let $\mathcal{X}$ be an arbitrary axiomatisation with axioms $A_{x_1}, ..., A_{x_n}$ and rules $R_{u_1}, ..., Ru_n$, where each rule is of the form "From $\phi_1, ..., \phi l$, infer ϕ_j". Then a derivation of a formula ϕ with $\mathcal{X}$ is a finite sequence $\phi_1, ..., \phi_m$ of formulas such that:
1) $\phi_m = \phi$ and
2)every ϕ_i in the sequence is:
a) either an instance of one of the axioms
b) or else the result of the aplication of one of the rules to formulas in the sequence that appear before ϕ_i

If there is a derivation for ϕ in $\mathcal{X}$, the we write $\vdash_{\mathcal{X}} \phi$ or $\vdash \phi$ if $\mathcal{X}$ is clear.
We say that ϕ is a theorem of $\mathcal{X}$.
Logic **K**is only (arbitrary) Kripke models, including models where R_i not not neccesarily reflect knowledge. E.g model $\mathcal{M}$

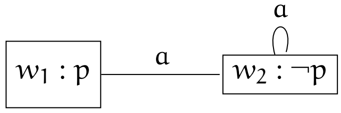


Figure 1: Model M_2

$(\mathcal{M}, w_1) \models p$ but,
$(\mathcal{M}, w_2)$ models$K_a \neg p$

We would like a logic where something like $\neg(p \wedge K_a \neg p)$ is a theorem.
Semantically, we solved this by requiring epistemic models to have reflexive accessibility relations (among other requirements).
Syntatically, add axiom $K_a\phi \rightarrow \phi$.
Additional axioms for $\mathcal{S}5$:
$K_a\phi \rightarrow \phi$ (T, truth)
$K_a\phi \rightarrow K_aK_a\phi$ (4, positive introspection)
$\neg K_a\phi \rightarrow K_a\neg K_a\phi$ (5, negative introspection)

Theorem 1 *Axiom system* ***K*** *is sound and complete w.r.t. the class* $\mathcal{K}$ *of all Kripke models, i.e. for every formula* ϕ *in* $\mathcal{L}_K$*, we have that* $\vdash_{\boldsymbol{K}} \phi$ *iff* $\mathcal{K} \models \phi$

Similarly, $\vdash_{\mathcal{S}5} \phi$ iff $\mathcal{S}5 \models \phi$. ("you can derive ϕ in $\mathcal{S}5$ iff ϕ is valid in all epistemic Kripke models")

2.4 Common knowledge

Group notions of knowledge:
Recall $E_B\phi$. E_B satisifes axiom T, but not positive introspection.
$E_B\phi \rightarrow E_BE_B\phi$ is not valid. E.g if agents a and b are both (separately) told that p is true, E_abp is true but not E_abE_abp.
So, how to model that everybody knows that everybody knows that... that p?
The common knowledge operator!
For $B \subseteq$, $C_B\phi \equiv \bigwedge_{n=0} E_B^n\phi$, where $E_B^n\phi = E_BE_B...E_B\phi$.

Definition 2 *By language* $\mathcal{L}_{KC}$*, we refer to the language defined just like* $\mathcal{L}_K$*, but with the additional C modality. For* $a \in A$*,* $B \subseteq A$*,* $p \in P$*, we define:*
$\phi ::= \phi|\neg\phi|\phi \wedge \phi|K_a\phi|C_B\phi$

Semantics: As before, using (epistemic) Kripke models.

Definition 3 *Let* $\mathcal{M} = \langle S, R, V\rangle$ *be a Kripke models with agents A and* $B \subseteq A$*. Then*
$R_{E_B} = \bigvee_{b \in B} R_b$
The transitive closure of a relation R is the smallest relation R^+ *s.t. :*
1-$R \subseteq R^+$
2-$\forall$ *x,y,z if* $(x, y) \in R^+$ *and* $(y, z) \in R^+$ *then also* $(x, z) \in R^+$
If additionally, $(x, x) \in R^+$ $\forall x$*, then* R^+ *is the reflexive-transitive closure of R,* R^*.

Definition 4 *Let P be a set of atomic propositions, A a set of agents and* $\mathcal{M} = \langle S, R, V\rangle$ *an epistemic model and* $B \subseteq A$*. Then the truth of an* $\mathcal{L}_K C$ *formula* ϕ *in (*$\mathcal{M}$*,s) is defined as for* $\mathcal{L}_K$*, with an additional clause for common knowledge.*
($\mathcal{M}$*s,)* $\models C_B\phi$ *iff* $(\mathcal{M}, t) \models \phi$ $\forall t \in S$ *with* $(s, t) \in \sim_{E_{B^*}}$ $(\sim_{C_B}) = R_{C_B})$

Example 1 $\mathcal{M}, w \models C_{ab} p$
$\mathcal{M}, w \not\models C_{abc} p$

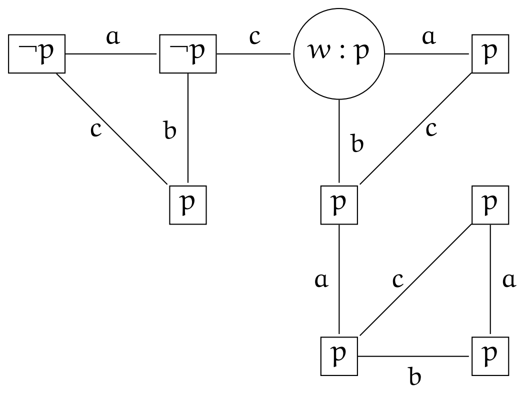


Figure 2: Example 1

Additional axioms for common knowledge:

$C_B(\phi \to \psi) \to (C_B\phi \to C_B\psi)$ (Dist)
$C_B\phi \to (\phi \wedge E_B C_B \phi)$ (Mix)
$C_B(\phi \to E_B\phi) \to (\phi \to C_B\phi)$ (Ind)
From ϕ, infer $C_B\phi$ (Nec)
Together with $\mathcal{S}5$ axioms and rules: sound and complete w.r.t. epistemic models with common knowledge.

2.5 Model checking

Local MC for $\mathcal{L}_K C$ formulas: Given a finite Kripke model $\mathcal{M} = \langle S, R, V \rangle$, an $\mathcal{L}_K C$ formulas ϕ and a state s, determine whether s satisfies ϕ:
you only care about state s. The rest of S may be given only implicitly.
Global MC for $\mathcal{L}_K C$ formulas: Given a finite Kripke model $\mathcal{M}_K C$, an $\mathcal{L}_K C$ formula ϕ, determine the set of states where ϕ is satisified.
We care about all states.
Especially easy if S is given explicitly.
Algorithmically often done semantically.

Idea: For all subformulas ψ of ϕ, determine the sets of states where ψ is true, inductively from small to large subformulas.

Definition 5 *Subformulas*
Let ϕ be a formula in the $\mathcal{L}_K C$ language. Then the set of subf(ϕ) of subformulas is

defined recursively as follows:
subf(p) = p for atomic propositions $p \in P$
subf($\neg\phi$) = {$\neg\phi$} $\cup$ *subf*(ϕ)
subf($\phi \vee \psi$) = {$\phi \vee \psi$} $\cup$ *subf*(ϕ) $\cup$ *subf*(ψ)
subf($K_a\phi$) = {$K_a\phi$} $\cup$ *subf*(ϕ)
subf($C_B\phi$) = {$C_B\phi$} $\cup$ *subf*(ϕ)

If $\psi \in subf(\phi) \setminus \{\phi\}$ then ψ is called a proper subformula of ϕ.

Definition 6 *Let a be an agent and* $S' \subseteq S$. *Then the strong preimage of S; w.r.t a is:*
$spreimg_a(S) = \{s \in S|$ *for* $s' \in S$ *with* $(s, s') \in R_a$*:* $s' \in S'\}$

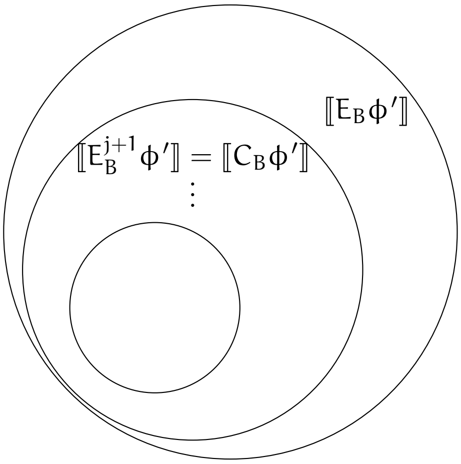
Notation:
Let $[\![\phi]\!] = \{s \in S | s \models \phi\}$ be the set of states where ϕ is true.

MC algorithm
Let $\mathcal{M} = \langle S, R, V\rangle$ be an (epistemic) Kripke model and $\phi \in \mathcal{L}_K C$ a formula. Let $\phi_1, ...\phi_n$ be the subformulas of ϕ ordered from small to large. Then:

Algorithm 1 Model checking

switch ϕ_i **do**
    case p
        $[\![p]\!] := V(p)$
    case $\neg\phi'$
        $[\![\phi_i]\!] := S \setminus [\![\phi']\!]$
    case $\phi' \vee \phi''$
        $[\![\phi_i]\!] := [\![\phi']\!] \cup [\![\phi'']\!]$
    case $\phi' \wedge \phi''$
        $[\![\phi_i]\!] := [\![\phi']\!] \cap [\![\phi'']\!]$
    case $K_a\phi'$
        $[\![\phi_i]\!] := spreimg_a([\![\phi']\!])$
    case $C_a\phi'$
        Let $S_1 = [\![\phi']\!]$
        Let $S_2 = S_1 \cap \bigcap_{b \in B} spreimg(S_1)$
        j:=1
        while $S_j \neq S_{j+1}$ **do**
            j:=j+1
            $S_{j+1} := S_j \cap \bigcap_{b \in B} spreimg(S_j)$
        end while
        Then $[\![\phi_i]\!] := S_{j+1}$

Intuition behind the $C_B\phi'$ case:



Example 2 $[\![\neg K_b(K_a p \wedge q)]\!]$ *?*
$[\![p]\!] = \{S_1, S_2, S_3, S_5, S_6\}$
$[\![q]\!] = \{S_2, S_3, S_4, S_5, S_6\}$
$[\![K_a p]\!] = \{S_1, S_2, S_3\}$
$[\![K_a \wedge q]\!] = \{S_2, S_3\}$
$[\![K_b(K_a p \wedge q)]\!] = \emptyset$
$[\![\neg K_b(K_a p \wedge q)]\!] = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

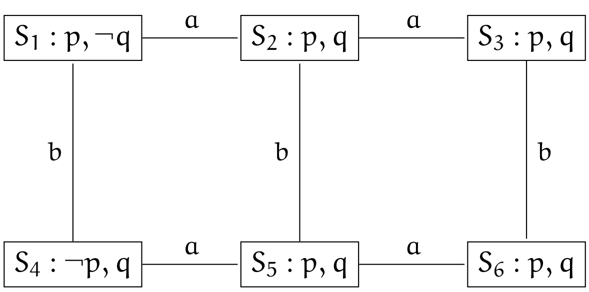


Figure 3: Example 2

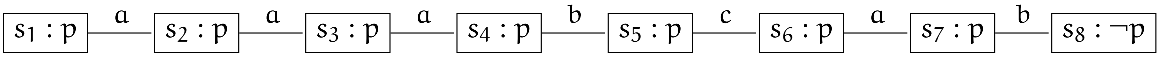


Figure 4: Example 3

Example 3 $[\![C_{ab}p]\!]$?

$[\![p]\!] = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = S_1$

$S_2 = S_1 \cap (spreimg_a(S_1) \cap spreimg_a(S_2))$

$= S_1 \cap (S_1 \cap \{s_1, .., s_6\}$

$= \{s_1, ..., s_6\}$

$S_3 = ... = \{s_1, ..., s_5\}$

$S_4 = S_3 = [\![C_{ab}p]\!] = \{s_1, ...s_5\}$