

# Principles of Knowledge Representation and Reasoning

## Nonmonotonic Reasoning

Bernhard Nebel, Stefan Wöfl, and Felix Lindner  
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## 1 Introduction

- Motivation
- Different forms of reasoning
- Different formalizations

### Introduction

Motivation  
Different forms of reasoning  
Different formalizations

### Default Logic

### Complexity

### Special Kinds of Defaults

### Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

3 / 48

## A reasoning task

- *If Mary has an essay to write, she will study late in the library.*
- *If the library is open, she will study late in the library.*
- *She has an essay to write.*

### Conclusion?

- *She will study late in the library.*

Reasoning tasks like this ([suppression task](#); Byrne, 1989) suggest that humans often do not reason as suggested by classical logics

### Introduction

Motivation  
Different forms of reasoning  
Different formalizations

### Default Logic

### Complexity

### Special Kinds of Defaults

### Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

4 / 48

## Nonmonotonic reasoning

How can we deal with the reasoning task given in the example?  
We can use a different representation that allows to restate the task as follows:

- *If Mary has an essay to write, she usually will study late in the library.*
- *She has an essay to write.*
- *If the library is not open, she will not study late in the library.*
- ...

### Introduction

Motivation  
Different forms of reasoning  
Different formalizations

### Default Logic

### Complexity

### Special Kinds of Defaults

### Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

5 / 48

# Nonmonotonic reasoning

- All logics presented so far are monotonic.
- A logic is called **monotonic** if all (logical) conclusions from a knowledge base remain justified when new information is added to the knowledge base.
- Cognitive studies indicate that everyday reasoning is often nonmonotonic (Stenning & Lambalgen, 2008; Johnson-Laird, 2010, etc.).
- When humans reason they use:
  - rules that may have **exceptions**:  
*If Mary has an essay to write, she **normally** will study late in the library.*
  - **default** assumptions:  
*The library is open.*

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

# Defaults in knowledge bases

Often we use **default** assumptions when definite information is not available or when we want to fix a standard value:

- 1 employee(anne)
- 2 employee(bert)
- 3 employee(carla)
- 4 employee(detlef)
- 5 employee(thomas)
- 6 onUnpaidMPaternityLeave(thomas)
- 7  $\text{employee}(X) \wedge \neg \text{onUnpaidMPaternityLeave}(X) \rightarrow \text{gettingSalary}(X)$
- 8 **Typically:**  $\text{employee}(X) \rightarrow \neg \text{onUnpaidMPaternityLeave}(X)$

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

# Defaults in common sense reasoning

- 1 **Tweety** is a **bird** like other birds.
  - 2 During the summer he stays in **Northern Europe**, in the winter he stays in **Africa**.
- Would you expect Tweety to be able to fly?
  - How does Tweety get from Northern Europe to Africa?

How would you formalize this in **formal logic** so that you get the expected answers?

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

# A formalization ...

- 1 bird(tweety)
  - 2  $\text{spend-summer}(\text{tweety}, \text{northern-europe}) \wedge \text{spend-winter}(\text{tweety}, \text{africa})$
  - 3  $\forall x(\text{bird}(x) \rightarrow \text{can-fly}(x))$
  - 4  $\text{far-away}(\text{northern-europe}, \text{africa})$
  - 5  $\forall xyz(\text{can-fly}(x) \wedge \text{far-away}(y, z) \wedge \text{spend-summer}(x, y) \wedge \text{spend-winter}(x, z) \rightarrow \text{flies}(x, y, z))$
- **But:** The implication (3) is just a **reasonable assumption**.
  - What if Tweety is an **emu**?

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

## Examples of such reasoning patterns

**Closed world assumption:** Database of **ground atoms**. All ground atoms not present are **assumed** to be false.

**Negation as failure:** In PROLOG, **NOT(P)** means “*P* is not **provable**” instead of “*P* is **provably false**”.

**Non-strict inheritance:** An attribute value is **inherited** only if there is no more specialized information contradicting the attribute value.

**Reasoning about actions:** When reasoning about actions, it is usually assumed that a property **changes** only if it **has to change**, i.e., properties by default do not change.

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

## Default, defeasible, and nonmonotonic reasoning

**Default reasoning:** **Jump to a conclusion** if there is no information that contradicts the conclusion.

**Defeasible reasoning:** Reasoning based on assumptions that can turn out to be wrong: **conclusions are defeasible**. In particular, **default reasoning** is defeasible.

**Nonmonotonic reasoning:** In classical logic, the set of consequences **grows monotonically** with the set of premises. If reasoning is **defeasible**, then reasoning becomes **nonmonotonic**.

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

## Approaches to nonmonotonic reasoning

- **Consistency-based:** **Extend** classical theory by rules that test whether an assumption is consistent with existing beliefs

⇒ Nonmonotonic logics such as **DL** (default logic), **NMLP** (nonmonotonic logic programming)

- **Entailment-based on normal models:** Models are ordered by **normality**. Entailment is determined by considering the most normal models only.

⇒ **Circumscription**, **preferential** and **cumulative logics**

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

## NM Logic – Consistency-based

If  $\varphi$  **typically implies**  $\psi$ ,  $\varphi$  is given, and **it is consistent to assume**  $\psi$ , then **conclude**  $\psi$ .

1 **Typically**  $\text{bird}(x)$  **implies**  $\text{can-fly}(x)$

2  $\forall x(\text{emu}(x) \rightarrow \text{bird}(x))$

3  $\forall x(\text{emu}(x) \rightarrow \neg \text{can-fly}(x))$

4  $\text{bird}(\text{tweety})$

⇒  $\text{can-fly}(\text{tweety})$

5  $\dots + \text{emu}(\text{tweety})$

⇒  $\neg \text{can-fly}(\text{tweety})$

Introduction  
Motivation  
Different forms of reasoning  
Different formalizations  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature

## NM Logic – Normal models

If  $\varphi$  typically implies  $\psi$ , then the models satisfying  $\varphi \wedge \psi$  should be **more normal** than those satisfying  $\varphi \wedge \neg\psi$ .

**Similar idea:** try to **minimize** the interpretation of “**Ab**normality” predicates.

- 1  $\forall x(\text{bird}(x) \wedge \neg\text{Ab}(x) \rightarrow \text{can-fly}(x))$
- 2  $\forall x(\text{emu}(x) \rightarrow \text{bird}(x))$
- 3  $\forall x(\text{emu}(x) \rightarrow \neg\text{can-fly}(x))$
- 4  $\text{bird}(\text{tweety})$

**Minimize interpretation of Ab:**

$\Rightarrow \text{can-fly}(\text{tweety})$

- 5 ... +  $\text{emu}(\text{tweety})$

$\Rightarrow$  Now in all models (incl. the normal ones):  $\neg \text{can-fly}(\text{tweety})$

Introduction

Motivation  
Different forms of reasoning

Different formalizations

Default Logic

Complexity

Special Kinds of Defaults

Literature

December 14 & 16, 2015

Nebel, Wölfl, Lindner – KR&R

14 / 48

UNI  
FREIBURG

## 2 Default Logic

- Basics
- Extensions
- Properties of extensions
- Normal defaults
- Default proofs
- Decidability

Introduction

Default Logic

Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds of Defaults

Literature

December 14 & 16, 2015

Nebel, Wölfl, Lindner – KR&R

16 / 48

UNI  
FREIBURG

## Default Logic – Outline

### 1 Introduction

### 2 Default Logic

- Basics
- Extensions
- Properties of extensions
- Normal defaults
- Default proofs
- Decidability

### 3 Complexity of Default Logic

### 4 Special Kinds of Defaults

Introduction

Default Logic

Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds of Defaults

Literature

December 14 & 16, 2015

Nebel, Wölfl, Lindner – KR&R

17 / 48

UNI  
FREIBURG

## Reiter's default logic: motivation

- We want to express something like “typically birds fly”.
- Add **non-logical inference rule**

$$\frac{\text{bird}(x) : \text{can-fly}(x)}{\text{can-fly}(x)}$$

with the **intended meaning**:

*If  $x$  is a bird and if it is consistent to assume that  $x$  can fly, then conclude that  $x$  can fly.*

- **Exceptions** can be represented as formulae:

$$\forall x(\text{penguin}(x) \rightarrow \neg\text{can-fly}(x))$$

$$\forall x(\text{emu}(x) \rightarrow \neg\text{can-fly}(x))$$

$$\forall x(\text{kiwi}(x) \rightarrow \neg\text{can-fly}(x))$$

Introduction

Default Logic

Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds of Defaults

Literature

December 14 & 16, 2015

Nebel, Wölfl, Lindner – KR&R

18 / 48

UNI  
FREIBURG

## Formal framework

- **FOL** with classical provability relation  $\vdash$  and deductive closure:  $\text{Th}(\Phi) := \{\varphi \mid \Phi \vdash \varphi\}$
- **Default rules:**  $\frac{\alpha : \beta}{\gamma}$ 
  - $\alpha$ : **Prerequisite**: must have been derived before rule can be applied.
  - $\beta$ : **Consistency condition**: the negation may not be derivable.
  - $\gamma$ : **Consequence**: will be concluded.
- A default rule is **closed** if it does not contain free variables.
- **(Closed) default theory**: A pair  $\langle D, W \rangle$ , where  $D$  is a countable set of (closed) default rules and  $W$  is a countable set of FOL formulae.

Introduction

Default Logic

Basics

Extensions  
Properties of extensions

Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds  
of Defaults

Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

19 / 48



## Extensions of default theories

Default theories **extend** the theory given by  $W$  using the default rules in  $D$  ( $\rightsquigarrow$  **extensions**). There may be zero, one, or many extensions.

### Example

$$W = \{a, \neg b \vee \neg c\}$$

$$D = \left\{ \frac{a : b}{b}, \frac{a : c}{c} \right\}$$

One **extension** contains  $b$ , the other contains  $c$ .

**Intuitively**, an **extension** is a set of **beliefs** resulting from  $W$  and  $D$ .

Introduction

Default Logic

Basics

Extensions  
Properties of extensions

Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds  
of Defaults

Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

20 / 48



## Decision problems about extensions in default logic

**Existence of extensions**: Does a default theory have an extension?

**Credulous reasoning**: If  $\varphi$  is in at least one extension,  $\varphi$  is a **credulous default conclusion**.

**Skeptical reasoning**: If  $\varphi$  is in all extensions,  $\varphi$  is a **skeptical default conclusion**.

Introduction

Default Logic

Basics

Extensions  
Properties of extensions

Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds  
of Defaults

Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

21 / 48



## Extensions (informally)

Desirable properties of an **extension**  $E$  of  $\langle D, W \rangle$ :

- 1 Contains all facts:  $W \subseteq E$ .
- 2 Is deductively closed:  $E = \text{Th}(E)$ .
- 3 All applicable default rules have been applied:  
**If**
  - 1  $(\frac{\alpha : \beta}{\gamma}) \in D$ ,
  - 2  $\alpha \in E$ ,
  - 3  $\neg \beta \notin E$**then**  $\gamma \in E$ .

- Further requirement: Application of default rules must follow in sequence (**groundedness**).

Introduction

Default Logic

Basics

Extensions  
Properties of extensions

Normal defaults  
Default proofs  
Decidability

Complexity

Special Kinds  
of Defaults

Literature

December 14 & 16, 2015

Nebel, Wöfl, Lindner – KR&R

22 / 48



# Groundedness

## Example

$$W = \emptyset$$

$$D = \left\{ \frac{a:b}{b}, \frac{b:a}{a} \right\}$$

**Question:** Should  $\text{Th}(\{a, b\})$  be an extension?

**Answer:** No!

$a$  can only be derived if we already have derived  $b$ .

$b$  can only be derived if we already have derived  $a$ .

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Extensions (formally)

## Definition

Let  $\Delta = \langle D, W \rangle$  be a closed default theory.

Let  $E$  be any set of closed formulae.

**Define:**

$$E_0 = W$$

$$E_i = \text{Th}(E_{i-1}) \cup \left\{ \gamma \mid \frac{\alpha:\beta}{\gamma} \in D, \alpha \in E_{i-1}, \neg\beta \notin E \right\}$$

$E$  is called an **extension** of  $\Delta$  if

$$E = \bigcup_{i=0}^{\infty} E_i.$$

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# How to use this definition?

- The definition does not tell us how to **construct** an extension.
- However, it tells us how to **check** whether a set is an extension:
  - 1 Guess a set  $E$ .
  - 2 Then construct sets  $E_i$  by starting with  $W$ .
  - 3 If  $E = \bigcup_{i=0}^{\infty} E_i$ , then  $E$  is an **extension** of  $\langle D, W \rangle$ .

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Examples

$D = \left\{ \frac{a:b}{b}, \frac{b:a}{a} \right\}$	$W = \{a \vee b\}$
$D = \left\{ \frac{a:b}{\neg b} \right\}$	$W = \emptyset$
$D = \left\{ \frac{a:b}{\neg b} \right\}$	$W = \{a\}$
$D = \left\{ \frac{a:b}{a}, \frac{b:c}{b}, \frac{c:c}{c} \right\}$	$W = \{b \rightarrow \neg a \wedge \neg c\}$
$D = \left\{ \frac{c:c}{\neg d}, \frac{d:d}{\neg e}, \frac{e:e}{\neg f} \right\}$	$W = \emptyset$
$D = \left\{ \frac{c:c}{\neg d}, \frac{d:d}{\neg c} \right\}$	$W = \emptyset$
$D = \left\{ \frac{a:b}{c}, \frac{a:d}{e} \right\}$	$W = \{a, \neg b \vee \neg d\}$

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Questions, questions, questions ...

- What can we say about the **existence** of extensions?
- How are the different extensions **related** to each other?
  - Can one extension be a **subset** of another one?
  - Are extensions **pairwise incompatible** (i.e. jointly inconsistent)?
- Can an extension be **inconsistent**?

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Properties of extensions: existence

## Theorem

- 1 If  $W$  is inconsistent, there is only one extension.
- 2 A closed default theory  $\langle D, W \rangle$  has an inconsistent extensions  $E$  if and only if  $W$  is inconsistent.

## Proof idea.

- 1 If  $W$  is inconsistent, no default rule is applicable and  $\text{Th}(W)$  is the only extension (which is inconsistent as well).
- 2 Claim 1  $\Rightarrow$  the **if**-part.  
For **only if**: Let  $W$  be consistent and assume that there exists an inconsistent extension  $E$ .  
Then there exists a consistent  $E_i$  such that  $E_{i+1}$  is inconsistent.  
That is, there is at least one applied default  $\alpha_i: \beta_i / \gamma_i$  with  $\gamma_i \in E_{i+1} \setminus \text{Th}(E_i)$ ,  $\alpha_i \in E_i$ , and  $\neg\beta_i \notin E$ .  
But this contradicts the inconsistency of  $E$ .

□

# Properties of extensions

## Theorem

If  $E$  and  $F$  are extensions of  $\langle D, W \rangle$  such that  $E \subseteq F$ , then  $E = F$ .

## Proof sketch.

$E = \bigcup_{i=0}^{\infty} E_i$  and  $F = \bigcup_{i=0}^{\infty} F_i$ . Use induction to show  $F_i \subseteq E_i$ .

Base case  $i = 0$ : Trivially  $E_0 = F_0 = W$ .

Inductive case  $i \geq 1$ : Assume  $\gamma \in F_{i+1}$ . Two cases:

- 1  $\gamma \in \text{Th}(F_i)$  implies  $\gamma \in \text{Th}(E_i)$  (because  $F_i \subseteq E_i$  by IH), and therefore  $\gamma \in E_{i+1}$ .
- 2 Otherwise  $\frac{\alpha: \beta}{\gamma} \in D$ ,  $\alpha \in F_i$ ,  $\neg\beta \notin F$ . However, then we have  $\alpha \in E_i$  (because  $F_i \subseteq E_i$ ) and  $\neg\beta \notin E$  (because of  $E \subseteq F$ ), i.e.,  $\gamma \in E_{i+1}$ .

□

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Normal default theories

All defaults in a **normal default theory** are **normal**:

$$\frac{\alpha: \beta}{\beta}.$$

## Theorem

Normal default theories have at least one extension.

## Proof sketch.

If  $W$  inconsistent, trivial.  
Otherwise construct

$$\begin{aligned} E_0 &= W \\ E_{i+1} &= \text{Th}(E_i) \cup T_i \quad E = \bigcup_{i=0}^{\infty} E_i \end{aligned}$$

where  $T_i$  is a maximal set s.t. (1)  $E_i \cup T_i$  is consistent and (2) if  $\beta \in T_i$  then there is  $\frac{\alpha: \beta}{\beta} \in D$  and  $\alpha \in E_i$ .

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Normal default theories: extensions are orthogonal

## Theorem (Orthogonality)

Let  $E$  and  $F$  be distinct extensions of a normal default theory. Then  $E \cup F$  is inconsistent.

### Proof.

Let  $E = \bigcup E_i$  and  $F = \bigcup F_i$  with

$$E_{i+1} = \text{Th}(E_i) \cup \left\{ \beta \mid \frac{\alpha : \beta}{\beta} \in D, \alpha \in E_i, \neg \beta \notin E \right\}$$

and the same for  $F$ .

Since  $E \neq F$ , there exists a smallest  $i$  such that  $E_{i+1} \neq F_{i+1}$ . This means there exists  $\frac{\alpha : \beta}{\beta} \in D$  with  $\alpha \in E_i = F_i$ , but with, say,  $\beta \in E_{i+1}$  and  $\beta \notin F_{i+1}$ . This is only possible if  $\neg \beta \in F$ .

This means,  $\beta \in E$  and  $\neg \beta \in F$ , i.e.,  $E \cup F$  is inconsistent.  $\square$

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Default proofs in normal default theories

## Definition

A **default proof** of  $\gamma$  in a normal default theory  $\langle D, W \rangle$  is a finite sequence of defaults  $(\delta_i = \frac{\alpha_i : \beta_i}{\beta_i})_{i=1, \dots, n}$  in  $D$  such that

- 1  $W \cup \{\beta_1, \dots, \beta_n\} \vdash \gamma$ ,
- 2  $W \cup \{\beta_1, \dots, \beta_n\}$  is consistent, and
- 3  $W \cup \{\beta_1, \dots, \beta_k\} \vdash \alpha_{k+1}$ , for  $0 \leq k \leq n-1$ .

## Theorem

Let  $\Delta = \langle D, W \rangle$  be a normal default theory so that  $W$  is consistent. Then  $\gamma$  has a default proof in  $\Delta$  if and only if there exists an extension  $E$  of  $\Delta$  such that  $\gamma \in E$ .

Test 2 (**consistency**) in the proof procedure suggests that default provability is not even **semi-decidable**.

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# Decidability

## Theorem

It is not semi-decidable to test whether a formula follows (skeptically or credulously) from a default theory.

### Proof.

Let  $\langle D, W \rangle$  be a default theory with  $W = \emptyset$  and  $D = \left\{ \frac{\cdot : \beta}{\beta} \right\}$  with  $\beta$  an arbitrary closed FOL formula. Clearly,  $\beta$  is in some/all extensions of  $\langle D, W \rangle$  if and only if  $\beta$  is satisfiable.

The existence of a semi-decision procedure for default proofs implies that there is a semi-decision procedure for satisfiability in FOL. But this is not possible because FOL validity is semi-decidable and this together with semi-decidability of FOL satisfiability would imply decidability of FOL, which is not the case.  $\square$

Introduction  
Default Logic  
Basics  
Extensions  
Properties of extensions  
Normal defaults  
Default proofs  
Decidability  
Complexity  
Special Kinds of Defaults  
Literature

# 3 Complexity of Default Logic

- Propositional DL
- Complexity of DL

Introduction  
Default Logic  
Complexity  
Propositional DL  
Complexity of DL  
Special Kinds of Defaults  
Literature



# Propositional default logic

Introduction  
Default Logic  
Complexity  
Propositional DL  
Complexity of DL  
Special Kinds of Defaults  
Literature

- **Propositional DL** is decidable.
- How difficult is reasoning in propositional DL?
- The **skeptical default reasoning** problem (does  $\varphi$  follow from  $\Delta$  skeptically:  $\Delta \mid\sim \varphi$ ?) is called **PDS**, credulous reasoning is called **LPDS**.
- PDS is **coNP-hard**:  
consider  $D = \emptyset$ ,  $W = \emptyset$
- LPDS is **NP-hard**:  
consider  $D = \left\{ \frac{:\beta}{\beta} \right\}$ ,  $W = \emptyset$ .

# Skeptical reasoning in propositional DL

Introduction  
Default Logic  
Complexity  
Propositional DL  
Complexity of DL  
Special Kinds of Defaults  
Literature

## Lemma

$PDS \in \Pi_2^P$ .

## Proof sketch.

We show that the complementary problem **UNPDS** (is there an extension  $E$  such that  $\varphi \notin E$ ) is in  $\Sigma_2^P$ .

The **algorithm**:

- 1 **Guess** set  $T \subseteq D$  of defaults, those that are applied.
- 2 **Verify** that defaults in  $T$  lead to  $E$ , using a **SAT oracle** and the guessed  $E := \text{Th} \left( \left\{ \gamma: \frac{\alpha:\beta}{\gamma} \in T \right\} \cup W \right)$ .
- 3 **Verify** that  $\left\{ \gamma: \frac{\alpha:\beta}{\gamma} \in T \right\} \cup W \not\models \varphi$  (**SAT oracle**).

$\leadsto \text{UNPDS} \in \Sigma_2^P$ . □

**Similar:**  $\text{LPDS} \in \Sigma_2^P$   
December 14 & 16, 2015

# $\Pi_2^P$ -Hardness

Introduction  
Default Logic  
Complexity  
Propositional DL  
Complexity of DL  
Special Kinds of Defaults  
Literature

## Lemma

$PDS$  is  $\Pi_2^P$ -hard.

## Proof sketch.

Reduction from 2QBF to UNPDS: For  $\exists \vec{a} \forall \vec{b} \varphi(\vec{a}, \vec{b})$  with  $\vec{a} = a_1, \dots, a_n$  and  $\vec{b} = b_1, \dots, b_m$  construct  $\Delta = \langle D, W \rangle$  with

$$D = \left\{ \frac{:\frac{a_i}{a_i}, \frac{:\neg a_i}{\neg a_i}, \frac{:\varphi(\vec{a}, \vec{b})}{\varphi(\vec{a}, \vec{b})}}{\frac{a_i}{a_i}, \neg a_i, \varphi(\vec{a}, \vec{b})} \right\}, \quad W = \emptyset$$

No extension contains both  $a_i$  and  $\neg a_i$ . Then:

$\Delta \mid\sim \neg \varphi(\vec{a}, \vec{b})$  iff there is an extension  $E$  s.t.  $\neg \varphi(\vec{a}, \vec{b}) \notin E$   
iff there is  $E$  s.t.  $\varphi(\vec{a}, \vec{b}) \in E$  (by  $\frac{:\varphi(\vec{a}, \vec{b})}{\varphi(\vec{a}, \vec{b})} \in D$ )  
iff there is  $A \subseteq \{a_1, \neg a_1, \dots, a_n, \neg a_n\}$  s.t.  $A \models \varphi(\vec{a}, \vec{b})$   
iff  $\exists \vec{a} \forall \vec{b} \varphi(\vec{a}, \vec{b})$  is true. □

# Conclusions & remarks

Introduction  
Default Logic  
Complexity  
Propositional DL  
Complexity of DL  
Special Kinds of Defaults  
Literature

## Theorem

$PDS$  is  $\Pi_2^P$ -complete, even for defaults of the form  $\frac{:\alpha}{\alpha}$ .

## Theorem

$LPDS$  is  $\Sigma_2^P$ -complete, even for defaults of the form  $\frac{:\alpha}{\alpha}$ .

- PDS is “**easier**” than reasoning in most modal logics.
- General and normal defaults have the same complexity.
- Polynomial special cases cannot be achieved by restricting, for example, to **Horn clauses** (satisfiability testing in polynomial time).
- It is necessary to restrict the underlying **monotonic reasoning problem** and the **number of extensions**.
- Similar results hold for other **nonmonotonic logics**.

## 4 Special Kinds of Defaults

- Semi-normal defaults
- Open defaults
- Outlook

Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Semi-normal defaults  
Open defaults  
Outlook  
Literature

## Semi-normal defaults (1)

Semi-normal defaults are sometimes useful:

$$\frac{\alpha : \beta \wedge \gamma}{\beta}$$

Important when one has **interacting** defaults:

$$\frac{\text{Adult}(x) : \text{Employed}(x)}{\text{Employed}(x)}$$

$$\frac{\text{Student}(x) : \text{Adult}(x)}{\text{Adult}(x)}$$

$$\frac{\text{Student}(x) : \neg \text{Employed}(x)}{\neg \text{Employed}(x)}$$

For **Student(TOM)** we get two extensions: one with **Employed(TOM)** and the other one with  $\neg \text{Employed}(\text{Tom})$ . Since the third rule is “**more specific**”, we may prefer it.

## Semi-normal defaults (2)

- Since being a student is an exception, we could use a **semi-normal** default to exclude students from employed adults:

$$\frac{\text{Student}(x) : \neg \text{Employed}(x)}{\neg \text{Employed}(x)}$$
$$\frac{\text{Adult}(x) : \text{Employed}(x) \wedge \neg \text{Student}(x)}{\text{Employed}(x)}$$
$$\frac{\text{Student}(x) : \text{Adult}(x)}{\text{Adult}(x)}$$

- Representing conflict-resolution by semi-normal defaults becomes clumsy when the number of default rules becomes high.
- A scheme for assigning **priorities** would be more elegant (there are indeed such schemes).

Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Semi-normal defaults  
Open defaults  
Outlook  
Literature

## Open defaults (1)

- Our examples included **open defaults**, but the theory covers only **closed defaults**.
- If we have  $\frac{\alpha(\vec{x}) : \beta(\vec{x})}{\gamma(\vec{x})}$ , then the variables should stand for all **nameable** objects.
- **Problem**: What about objects that have been introduced implicitly, e.g., via formulae such as  $\exists x P(x)$ .
- **Solution by Reiter**: Skolemization of all formulae in  $W$  and  $D$ .
- **Interpretation**: An open default stands for all the closed defaults resulting from substituting **ground terms** for the variables.

Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Semi-normal defaults  
Open defaults  
Outlook  
Literature

## Open defaults (2)

Skolemization can create problems because it preserves satisfiability, but it is not an equivalence transformation.

### Example

$$\begin{aligned} & \forall x(\text{Man}(x) \leftrightarrow \neg \text{Woman}(x)) \\ & \forall x(\text{Man}(x) \rightarrow (\exists y(\text{Spouse}(x, y) \wedge \text{Woman}(y)) \vee \text{Bachelor}(x))) \\ & \text{Man}(\text{TOM}) \\ & \text{Spouse}(\text{TOM}, \text{MARY}) \\ & \text{Woman}(\text{MARY}) \\ & \frac{\text{Man}(x)}{\text{Man}(x)} \end{aligned}$$

Skolemization of  $\exists y: \dots$  enables concluding [Bachelor\(TOM\)](#)! The reason is that for  $g(\text{TOM})$  we get  $\text{Man}(g(\text{TOM}))$  [by default](#) (where  $g$  is the Skolem function).

Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Semi-normal defaults  
Open defaults  
Outlook  
Literature

## Open defaults (3)

It is even worse: Logically equivalent theories can have different extensions:

$$\begin{aligned} W_1 &= \{ \exists x(P(c, x) \vee Q(c, x)) \} \\ W_2 &= \{ \exists x P(c, x) \vee \exists x Q(c, x) \} \\ D &= \left\{ \frac{P(x, y) \vee Q(x, y) : R}{R} \right\} \end{aligned}$$

$W_1$  and  $W_2$  are logically equivalent. However, the Skolemization of  $W_1$ , symbolically  $s(W_1)$ , is not equivalent with  $s(W_2)$ . The only extension of  $\langle D, W_1 \rangle$  is  $\text{Th}(s(W_1) \cup R)$ . The only extension of  $\langle D, W_2 \rangle$  is  $\text{Th}(s(W_2))$ .

[Note](#): Skolemization is not the right method to deal with open defaults in the general case.

Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Semi-normal defaults  
Open defaults  
Outlook  
Literature

## Outlook

Although Reiter's definition of DL makes sense, one can come up with a number of variations and extend the investigation ...

- Extensions can be defined differently (e.g., by remembering consistency conditions).
- ... or by removing the groundedness condition.
- Open defaults can be handled differently (more model-theoretically).
- General proof methods for the finite, decidable case
- Applications of default logic:
  - Diagnosis
  - Reasoning about actions

Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Semi-normal defaults  
Open defaults  
Outlook  
Literature

## Literature



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Introduction  
Default Logic  
Complexity  
Special Kinds of Defaults  
Literature