# Principles of Knowledge Representation and Reasoning

## Semantic Networks and Description Logics III: Description Logics – Reasoning Services and Reductions

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November 25, 2015

# Motivation

# Example TBox & ABox

$$
\begin{aligned}
\mathtt{Male} &\doteq \neg\mathtt{Female}\\
\mathtt{Human} &\sqsubseteq \mathtt{Living\_entity}\\
\mathtt{Woman} &\doteq \mathtt{Human} \sqcap \mathtt{Female}\\
\mathtt{Man} &\doteq \mathtt{Human} \sqcap \mathtt{Male}\\
\mathtt{Mother} &\doteq \mathtt{Woman} \sqcap \exists\mathtt{has\text{-}child.Human}\\
\mathtt{Father} &\doteq \mathtt{Man} \sqcap \exists\mathtt{has\text{-}child.Human}\\
\mathtt{Parent} &\doteq \mathtt{Father} \sqcup \mathtt{Mother}\\
\mathtt{Grandmother} &\\
&\doteq \mathtt{Woman} \sqcap \exists\mathtt{has\text{-}child.Parent}\\
\mathtt{Mother\text{-}without\text{-}daughter} &\\
&\doteq \mathtt{Mother} \sqcap \forall\mathtt{has\text{-}child.Male}\\
\mathtt{Mother\text{-}with\text{-}many\text{-}children} &\\
&\doteq \mathtt{Mother} \sqcap (\geq 3\,\mathtt{has\text{-}child})
\end{aligned}
$$

| | | |
|---|---|---|
| DIANA: | Woman | |
| ELIZABETH: | Woman | |
| CHARLES: | Man | |
| EDWARD: | Man | |
| ANDREW: | Man | |
| DIANA: | Mother-without-daughter | |
| (ELIZABETH, | CHARLES): | has-child |
| (ELIZABETH, | EDWARD): | has-child |
| (ELIZABETH, | ANDREW): | has-child |
| (DIANA, | WILLIAM): | has-child |
| (CHARLES, | WILLIAM): | has-child |

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- These problems can be reduced to logical satisfiability or implication – using the logical semantics.
- *However,* we take a different route: we will try to simplify these problems and then we specify direct inference methods.



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# Basic Reasoning Services





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# Satisfiability of concept descriptions



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## Example

$\mathtt{Woman} \sqcap (\leq 0\, \mathtt{has\text{-}child}) \sqcap (\geq 1\, \mathtt{has\text{-}child})$ is unsatisfiable.



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## Example

$\texttt{Mother-without-daughter} \sqcap \forall \texttt{has-child.Female}$ is unsatisfiable, given our previously specified family TBox.

# Reduction: Getting rid of the TBox

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Idea:

- Since TBoxes are cycle-free, one can understand a concept definition as a kind of "macro".
- For a given TBox $\mathcal{T}$ and a given concept description $C$, all defined concept symbols appearing in $C$ can be expanded until $C$ contains only undefined concept symbols.
- An expanded concept description is then satisfiable if and only if $C$ is satisfiable in $\mathcal{T}$.
- *Problem*: What do we do with partial definitions (using $\sqsubseteq$)?

# Normalized terminologies

- A terminology is called normalized when it does not contain definitions fo the form $A \sqsubseteq C$.
- In order to normalize a terminology, replace

$$A \sqsubseteq C$$

by

$$A \doteq A^* \sqcap C,$$

where $A^*$ is a fresh concept symbol (not appearing elsewhere in $\mathcal{T}$).
- If $\mathcal{T}$ is a terminology, the normalized terminology is denoted by $\widetilde{\mathcal{T}}$.





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# Normalizing is reasonable



## Theorem (Normalization invariance)

*If $\mathcal{I}$ is a model of the terminology $\mathcal{T}$, then there exists a model $\mathcal{I}'$ of $\widetilde{\mathcal{T}}$ such that for all concept symbols $A$ occurring in $\mathcal{T}$, it holds $A^{\mathcal{I}} = A^{\mathcal{I}'}$, and vice versa.*

## Proof.

"$\Rightarrow$": Let $\mathcal{I}$ be a model of $\mathcal{T}$. This model should be extended to $\mathcal{I}'$ so that the freshly introduced concept symbols also get interpretations.



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Then set $A^{*\mathcal{I}'} := A^{\mathcal{I}}$.
$\mathcal{I}'$ obviously satisfies $\widetilde{\mathcal{T}}$ and has the same interpretation for all symbols in $\mathcal{T}$.
"$\Leftarrow$": Given a model $\mathcal{I}'$ of $\widetilde{\mathcal{T}}$, its restriction to symbols of $\mathcal{T}$ is the interpretation we look for. $\square$

# TBox unfolding

- We say that a normalized TBox is unfolded by one step when all defined concept symbols on the right sides are replaced by their defining terms.
- Example: $\texttt{Mother} \doteq \texttt{Woman} \sqcap \ldots$ is unfolded to $\texttt{Mother} \doteq (\texttt{Human} \sqcap \texttt{Female}) \sqcap \ldots$

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- Example: $\texttt{Mother} \doteq \texttt{Woman} \sqcap \ldots$ is unfolded to $\texttt{Mother} \doteq (\texttt{Human} \sqcap \texttt{Female}) \sqcap \ldots$
- We write $U(\mathcal{T})$ to denote a one-step unfolding and $U^n(\mathcal{T})$ to denote an $n$-step unfolding.
- We say that $\mathcal{T}$ is unfolded if $U(\mathcal{T}) = \mathcal{T}$.
- $U^n(\mathcal{T})$ is called the unfolding of $\mathcal{T}$ if $U^n(\mathcal{T}) = U^{n+1}(\mathcal{T})$. If such an unfolding exists, it is denoted by $\widehat{\mathcal{T}}$.

# Properties of unfoldings (1): Existence



**Theorem (Existence of unfolded terminology)**

*Each normalized terminology $\mathcal{T}$ can be unfolded, i.e., its unfolding $\widehat{\mathcal{T}}$ exists.*

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**Proof idea.**

The main reason is that terminologies have to be cycle-free. The proof can be done by induction of the definition depth of concepts. □

# Properties of unfoldings (2): Equivalence

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"$\Rightarrow$": Let $\mathcal{I}$ be a model of $\mathcal{T}$.

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"$\Rightarrow$": Let $\mathcal{I}$ be a model of $\mathcal{T}$.
Then it is also a model of $U(\mathcal{T})$, since on the right side of the definitions only terms with identical interpretations are substituted.

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"$\Leftarrow$": Let $\mathcal{I}$ be a model for $U(\mathcal{T})$. Clearly, this is also a model of $\mathcal{T}$ (with the same argument as above).

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"$\Leftarrow$": Let $\mathcal{I}$ be a model for $U(\mathcal{T})$. Clearly, this is also a model of $\mathcal{T}$ (with the same argument as above).
This means that any model $\widehat{\mathcal{T}}$ is also a model of $\mathcal{T}$. $\square$

# Generating models

- All concept and role names not occurring on the left hand side of definitions in a terminology $\mathcal{T}$ are called primitive components.
- Interpretations restricted to primitive components are called initial interpretations.

**Theorem (Model extension)**

*For each initial interpretation $\mathcal{J}$ of a normalized TBox, there exists a unique interpretation $\mathcal{I}$ extending $\mathcal{J}$ and satisfying $\mathcal{T}$.*

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**Corollary (Model existence for TBoxes)**

*Each TBox has at least one model.*

# Unfolding of concept descriptions

- Similar to the unfolding of TBoxes, we can define the unfolding of a concept description.
- We write $\widehat{C}$ for the unfolded version of $C$.

**Theorem (Satisfiability of unfolded concepts)**

*An concept description $C$ is satisfiable in a terminology $\mathcal{T}$ if and only if $\widehat{C}$ satisfiable in an empty terminology.*

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Then extend it to a full model $\mathcal{I}$ of $\mathcal{T}$.
This satisfies $\mathcal{T}$ as well as $\widehat{C}$. Since $\widehat{C}^{\mathcal{I}} = C^{\mathcal{I}}$, it satisfies also $C$. $\square$

# General TBox Reasoning Services

# Subsumption in a TBox

## Subsumption in a TBox

Given a terminology $\mathcal{T}$ and two concept descriptions $C$ and $D$, is $C$ subsumed by (or a sub-concept of) $D$ in $\mathcal{T}$ (symb. $C \sqsubseteq_{\mathcal{T}} D$)?

Test:

- Is $C$ interpreted as a subset of $D$ in each model $\mathcal{I}$ of $\mathcal{T}$, i.e. $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$?
- Is the formula $\forall x \big(C(x) \rightarrow D(x)\big)$ a logical consequence of the translation of $\mathcal{T}$ into FOL?

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## Example

Given our family TBox, it holds $\texttt{Grandmother} \sqsubseteq_{\mathcal{T}} \texttt{Mother}$.



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# Subsumption (without a TBox)

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Given two concept descriptions $C$ and $D$, is $C$ subsumed by $D$ regardless of a TBox (or in an empty TBox) (symb. $C \sqsubseteq D$)?

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## Example

Clearly, $\texttt{Human} \sqcap \texttt{Female} \sqsubseteq \texttt{Human}$.





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# Reductions

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- Subsumption in the empty TBox can be reduced to unsatisfiability:
  … $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable.
- Unsatisfiability can be reduced to subsumption:
  … $C$ is unsatisfiable iff $C \sqsubseteq (C \sqcap \neg C)$.

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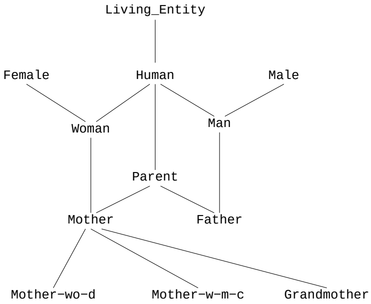
Useful in order to:

- check the modeling
- use the precomputed relations later when subsumption queries have to be answered

Problem can be reduced to subsumption checking: then it is a generalized sorting problem!

### Example



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## Example

The ABox

$$X : (\forall r.\neg C), \quad Y : C, \quad (X, Y) : r$$

is not satisfiable.

# ABox satisfiability in a TBox

## ABox satisfiability in a TBox

Given an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, is $\mathcal{A}$ consistent with the terminology introduced in $\mathcal{T}$, i.e., is $\mathcal{T} \cup \mathcal{A}$ satisfiable?

## Example

If we extend our example with

```
MARGRET: Woman
(DIANA,MARGRET): has-child,
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then the ABox becomes unsatisfiable in the given TBox.

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then the ABox becomes unsatisfiable in the given TBox.

- Problem is reducible to satisfiability of an ABox: … normalize terminology, then unfold all concept and role descriptions in the ABox

# Instance relations

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Which additional ABox formulae of the form $a\colon C$ follow logically from a given ABox and TBox?

- Is $a^{\mathcal{I}} \in C^{\mathcal{I}}$ true in all models $\mathcal{I}$ of $\mathcal{T} \cup \mathcal{A}$?
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Reductions:

- Instance relations wrt. an ABox and a TBox can be reduced to instance relations wrt. ABox: use normalization and unfolding
- Instance relations in an ABox can be reduced to ABox unsatisfiability:

$$a\colon C \text{ holds in } \mathcal{A} \iff \mathcal{A} \cup \{a\colon \neg C\} \text{ is unsatisfiable}$$

# Examples

## Example

- ELIZABETH: Mother-with-many-children?

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  yes

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- ELIZABETH: Mother-with-many-children?
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- $\mathtt{ELIZABETH}$: $\mathtt{Mother\text{-}with\text{-}many\text{-}children}$?
  yes
- $\mathtt{WILLIAM}$: $\neg\,\mathtt{Female}$?
  yes
- $\mathtt{ELIZABETH}$: $\mathtt{Mother\text{-}without\text{-}daughter}$?
  no (no CWA!)

# Examples

## Example

- ELIZABETH: Mother-with-many-children?
  yes
- WILLIAM: ¬ Female?
  yes
- ELIZABETH: Mother-without-daughter?
  no (no CWA!)
- ELIZABETH: Grandmother?

# Examples

## Example

- ELIZABETH: $\texttt{Mother-with-many-children}$?

  yes

- WILLIAM: $\neg$ $\texttt{Female}$?

  yes

- ELIZABETH: $\texttt{Mother-without-daughter}$?

  no (no CWA!)

- ELIZABETH: $\texttt{Grandmother}$?

  no (only male, but not necessarily human!)

# Realization



## Realization

For a given object *a*, determine the most specialized concept symbols such that *a* is an instance of these concepts

### Motivation:

- Similar to classification
- Is the minimal representation of the instance relations (in the set of concept symbols)
- Will give us faster answers for instance queries!

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For a given object *a*, determine the most specialized concept symbols such that *a* is an instance of these concepts

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- Similar to classification
- Is the minimal representation of the instance relations (in the set of concept symbols)
- Will give us faster answers for instance queries!

Reduction: Can be reduced to (a sequence of) instance relation tests.



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# Retrieval

## Retrieval

Given a concept description $C$, determine the set of all (specified) instances of the concept description.

## Example

We ask for all instances of the concept `Male`.
For our TBOX/ABox we will get the answer `CHARLES`, `ANDREW`, `EDWARD`, `WILLIAM`.

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## Retrieval

Given a concept description $C$, determine the set of all (specified) instances of the concept description.

## Example

We ask for all instances of the concept `Male`.
For our TBOX/ABox we will get the answer `CHARLES`, `ANDREW`, `EDWARD`, `WILLIAM`.

- Reduction: Compute the set of instances by testing the instance relation for each object!
- Implementation: Realization can be used to speed this up

# Summary and Outlook

# Reasoning services – summary

- Satisfiability of concept descriptions
  - in a given TBox or in an empty TBox
- Subsumption between concept descriptions
  - in a given TBox or in an empty TBox
- Classification
- Satisfiability of an ABox
  - in a given TBox or in an empty TBox
- Instance relations in an ABox
  - in a given TBox or in an empty TBox
- Realization
- Retrieval



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# Outlook

- How to determine subsumption between two concept descriptions (in the empty TBox)?
- How to determine instance relations/ABox satisfiability?
- How to implement the mentioned reductions efficiently?
- Does normalization and unfolding introduce another source of computational complexity?