Principles of Knowledge Representation and Reasoning

Semantic Networks and Description Logics II: Description Logics – Terminology and Notation

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November 23, 2015

Introduction

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Motivation

- Main problem with semantic networks and frames
  … the lack of formal semantics!
- Disadvantage of simple inheritance networks
  … concepts are atomic and do not have any structure

$\leadsto$ Brachman's structural inheritance networks (1977)

Structural inheritance networks

- Concepts are defined/described using a small set of well-defined operators
- Distinction between conceptual and object-related knowledge
- Computation of subconcept relation and of instance relation
- Strict inheritance (of the entire structure of a concept): inherited properties cannot be overriden

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Systems and applications

- Systems:
  - KL-ONE: First implementation of the ideas (1978)
  - then: NIKL, KL-TWO, KRYPTON, KANDOR, CLASSIC, BACK, KRIS, YAK, CRACK ...
  - later: FaCT, DLP, RACER 1998
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  - and information systems,
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- Applications:
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- Languages: DAML+OIL, now OWL (Web Ontology Language)

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Description logics

- Previously also known as KL-ONE-alike languages, frame-based languages, terminological logics, concept languages

Description logics

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- Description Logics (DL) allow us
  - to describe concepts using complex descriptions,
  - to introduce the terminology of an application and to structure it (TBox),
  - to introduce objects and relate them to the introduced terminology (ABox),
  - and to reason about the terminology and the objects.

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Informal example

`Male` is:	the opposite of `female`
A `human` is a kind of:	`living entity`
A `woman` is:	a `human` and a `female`
A `man` is:	a `human` and a `male`
A `mother` is:	a `woman` with at least one `child` that is a `human`
A `father` is:	a `man` with at least one `child` that is a `human`
A `parent` is:	a `mother` or a `father`
A `grandmother` is:	a `woman`, with at least one `child` that is a `parent`
A `mother-wod` is:	a `mother` with only `male` `children`

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`Elizabeth` is a `woman`
`Elizabeth` has the child `Charles`
`Charles` is a `man`
`Diana` is a `mother-wod`
`Diana` has the child `William`

Possible Questions :
Is a `grandmother` a `parent`?
Is `Diana` a `parent`?
Is `William` a `man`?
Is `Elizabeth` a `mother-wod`?

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Concepts and Roles

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Atomic concepts and roles

- Concept names:
  - E.g., `Grandmother`, `Male`, ... (in the following usually capitalized)
  - We will use symbols such as $A, A_1, \ldots$ for concept names
  - Semantics: Monadic predicates $A(\cdot)$ or set-theoretically a subset of the universe $A^{\mathcal{I}} \subseteq \mathcal{D}$.

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  - Semantics: Monadic predicates $A(\cdot)$ or set-theoretically a subset of the universe $A^{\mathcal{I}} \subseteq \mathcal{D}$.
- Role names:
  - In our example, e.g., `child`. Often we will use names such as `has-child` or something similar (in the following usually lowercase).
  - Role names are disjoint from concept names
  - Symbolically: $t, t_1, \ldots$
  - Semantics: Binary relations $t(\cdot,\cdot)$ or set-theoretically $t^{\mathcal{I}} \subseteq \mathcal{D} \times \mathcal{D}$.

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Concept and role description

- From (atomic) concept and role names, complex concept and role descriptions can be created
- In our example, e.g., "`Human` <u>and</u> `Female`."
- Symbolically: C for concept descriptions and r for role descriptions

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Which particular constructs are available depends on the chosen description logic!

- FOL semantics: A concept description C corresponds to a formula $C(x)$ with the free variable x.
  Similarly with role descriptions r: they correspond to formulae $r(x,y)$ with free variables x,y.
- Set semantics:

$$C^{\mathcal{I}} = \{d \in \mathcal{D} : C(d) \text{ "is true in" } \mathcal{I}\}$$

$$r^{\mathcal{I}} = \{(d,e) \in \mathcal{D}^2 : r(d,e) \text{ "is true in" } \mathcal{I}\}$$

Boolean operators

- Syntax: let C and D be concept descriptions, then the following are also concept descriptions:
  - $C \sqcap D$ (concept conjunction)
  - $C \sqcup D$ (concept disjunction)
  - $\neg C$ (concept negation)

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- Examples:
  - $\texttt{Human} \sqcap \texttt{Female}$
  - $\texttt{Father} \sqcup \texttt{Mother}$
  - $\neg\,\texttt{Female}$

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- FOL semantics: $C(x) \land D(x)$, $C(x) \lor D(x)$, $\neg C(x)$

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- Examples:
  - $\texttt{Human} \sqcap \texttt{Female}$
  - $\texttt{Father} \sqcup \texttt{Mother}$
  - $\neg\,\texttt{Female}$
- FOL semantics: $C(x) \wedge D(x)$, $C(x) \vee D(x)$, $\neg C(x)$
- Set semantics: $C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $\mathcal{D} \setminus C^{\mathcal{I}}$

Role restrictions

- Motivation:
  - Often we want to describe something by restricting the possible “fillers” of a role, e.g. `Mother-wod`.
  - Sometimes we want to say that there is at least a filler of a particular type, e.g. `Grandmother`

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- Idea: Use quantifiers that range over the role-fillers
  - $\mathtt{Mother} \sqcap \forall \mathtt{has\text{-}child.Man}$
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- FOL semantics:

$$(\exists r.C)(x) = \exists y(r(x,y) \wedge C(y))$$
$$(\forall r.C)(x) = \forall y\,(r(x,y) \rightarrow C(y))$$

- Set semantics:

$$(\exists r.C)^{\mathcal{I}} = \left\{d \in \mathcal{D} : \text{ there ex. some } e \text{ s.t. } (d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}}\right\}$$
$$(\forall r.C)^{\mathcal{I}} = \left\{d \in \mathcal{D} : \text{ for each } e \text{ with } (d,e) \in r^{\mathcal{I}},\ e \in C^{\mathcal{I}}\right\}$$

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Cardinality restriction

- Motivation:
  - Often we want to describe something by restricting the number of possible “fillers” of a role, e.g., a `Mother` with at least 3 `children` or at most 2 `children`.
- Idea: We restrict the cardinality of the role filler sets:
  - $\texttt{Mother} \sqcap \geq 3\,\texttt{has-child}$
  - $\texttt{Mother} \sqcap \leq 2\,\texttt{has-child}$

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  - $\texttt{Mother} \sqcap \geq 3\, \texttt{has-child}$
  - $\texttt{Mother} \sqcap \leq 2\, \texttt{has-child}$
- FOL semantics:

$$(\geq n\ r)(x) = \exists y_1 \ldots y_n \Big(r(x, y_1) \land \cdots \land r(x, y_n) \land y_1 \neq y_2 \land \cdots \land y_{n-1} \neq y_n \Big)$$

$$(\leq n\ r)(x) = \neg(\geq n+1\ r)(x)$$

- Set semantics:

$$(\geq n\ r)^{\mathcal{I}} = \Big\{ d \in \mathcal{D} : \Big| \Big\{ e \in \mathcal{D} : r^{\mathcal{I}}(d, e) \Big\} \Big| \geq n \Big\}$$

$$(\leq n\ r)^{\mathcal{I}} = \mathcal{D} \setminus (\geq n+1\ r)^{\mathcal{I}}$$

Inverse roles

- Motivation:
  - How can we describe the concept "children of rich parents"?
- Idea: Define the "inverse" role for a given role (the converse relation)
  - $\texttt{has-child}^{-1}$
- Example: $\exists\texttt{has-child}^{-1}.\texttt{Rich}$

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  - $\texttt{has-child}^{-1}$
- Example: $\exists\texttt{has-child}^{-1}.\texttt{Rich}$
- FOL semantics:

$$r^{-1}(x,y) = r(y,x)$$

- Set semantics:

$$(r^{-1})^{\mathcal{I}} = \left\{(d,e) \in \mathcal{D}^2 : (e,d) \in r^{\mathcal{I}}\right\}$$

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Role composition

- Motivation:
  - How can we define the role `has-grandchild` given the role `has-child`?
- Idea: Compose roles (as one can compose binary relations)
  - `has-child` ∘ `has-child`

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Role composition

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  - How can we define the role `has-grandchild` given the role `has-child`?
- Idea: Compose roles (as one can compose binary relations)
  - `has-child` $\circ$ `has-child`
- FOL semantics:

$$(r \circ s)(x,y) = \exists z(r(x,z) \wedge s(z,y))$$

- Set semantics:

$$(r \circ s)^{\mathcal{I}} = \left\{(d,e) \in \mathcal{D}^2 : \exists f \text{ s.t. } (d,f) \in r^{\mathcal{I}} \wedge (f,e) \in s^{\mathcal{I}}\right\}$$

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Role value maps

- Motivation:
  - How do we express the concept "women who know all the friends of their children"
- Idea: Relate role filler sets to each other
  - $\texttt{Woman} \sqcap (\texttt{has-child} \circ \texttt{has-friend} \sqsubseteq \texttt{knows})$
- FOL semantics:

$$(r \sqsubseteq s)(x) = \forall y \big(r(x,y) \rightarrow s(x,y)\big)$$

- Set semantics: Let $r^{\mathcal{I}}(d) = \big\{e : r^{\mathcal{I}}(d,e)\big\}$.

$$(r \sqsubseteq s)^{\mathcal{I}} = \big\{d \in \mathcal{D} : r^{\mathcal{I}}(d) \subseteq s^{\mathcal{I}}(d)\big\}$$

- *Note*: Role value maps lead to undecidability of satisfiability testing of concept descriptions!

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TBox and ABox

Terminology box

- In order to introduce new terms, we use two kinds of terminological axioms:
  - $A \doteq C$
  - $A \sqsubseteq C$

  where A is a concept name and C is a concept description.

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Terminology box

- In order to introduce new terms, we use two kinds of terminological axioms:
  - $A \doteq C$
  - $A \sqsubseteq C$

  where A is a concept name and C is a concept description.
- A terminology or TBox is a finite set of such axioms with the following additional restrictions:
  - no multiple definitions of the same symbol such as $A \doteq C$, $A \sqsubseteq D$
  - no cyclic definitions (even not indirectly), such as $A \doteq \forall r . B$, $B \doteq \exists s . A$

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TBoxes: semantics

- TBoxes restrict the set of possible interpretations.
- FOL semantics:
  - $A \doteq C$ corresponds to $\forall x \big(A(x) \leftrightarrow C(x)\big)$
  - $A \sqsubseteq C$ corresponds to $\forall x \big(A(x) \rightarrow C(x)\big)$

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- Set semantics:
  - $A \doteq C$ corresponds to $A^{\mathcal{I}} = C^{\mathcal{I}}$
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- Set semantics:
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  - $A \sqsubseteq C$ corresponds to $A^{\mathcal{I}} \subseteq C^{\mathcal{I}}$
- Non-empty interpretations which satisfy all terminological axioms are called models of the TBox.

Assertional box

- In order to state something about objects in the world, we use two forms of assertions:
  - $a : C$
  - $(a, b) : r$

  where a and b are individual names (e.g., ELIZABETH, PHILIP), C is a concept description, and r is a role description.
- An ABox is a finite set of assertions.

ABoxes: semantics

- Individual names are interpreted as elements of the universe under the unique-name-assumption, i.e., different names refer to different objects.
- Assertions express that an object is an instance of a concept or that two objects are related by a role.

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ABoxes: semantics

- Individual names are interpreted as elements of the universe under the unique-name-assumption, i.e., different names refer to different objects.
- Assertions express that an object is an instance of a concept or that two objects are related by a role.
- FOL semantics:
  - $a : C$ corresponds to $C(a)$
  - $(a,b) : r$ corresponds to $r(a,b)$
- Set semantics:
  - $a^{\mathcal{I}} \in D$
  - $a : C$ corresponds to $a^{\mathcal{I}} \in C^{\mathcal{I}}$
  - $(a,b) : r$ corresponds to $(a^{\mathcal{I}},b^{\mathcal{I}}) \in r^{\mathcal{I}}$

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  - $(a,b) : r$ corresponds to $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$
- Models of an ABox and of ABox + TBox can be defined analogously to models of a TBox.

Example TBox

$$
\begin{array}{rcl}
\texttt{Male} & \doteq & \neg\texttt{Female} \\
\texttt{Human} & \sqsubseteq & \texttt{Living\_entity} \\
\texttt{Woman} & \doteq & \texttt{Human} \sqcap \texttt{Female} \\
\texttt{Man} & \doteq & \texttt{Human} \sqcap \texttt{Male} \\
\texttt{Mother} & \doteq & \texttt{Woman} \sqcap \exists\texttt{has-child.Human} \\
\texttt{Father} & \doteq & \texttt{Man} \sqcap \exists\texttt{has-child.Human} \\
\texttt{Parent} & \doteq & \texttt{Father} \sqcup \texttt{Mother} \\
\texttt{Grandmother} & \doteq & \texttt{Woman} \sqcap \exists\texttt{has-child.Parent} \\
\texttt{Mother-without-daughter} & \doteq & \texttt{Mother} \sqcap \forall\texttt{has-child.Male} \\
\texttt{Mother-with-many-children} & \doteq & \texttt{Mother} \sqcap (\geq 3\,\texttt{has-child})
\end{array}
$$

Example ABox

CHARLES:	Man	DIANA: Woman
EDWARD:	Man	ELIZABETH: Woman
ANDREW:	Man	
DIANA:	Mother-without-daughter	

(ELIZABETH, CHARLES):	has-child
(ELIZABETH, EDWARD):	has-child
(ELIZABETH, ANDREW):	has-child
(DIANA, WILLIAM):	has-child
(CHARLES, WILLIAM):	has-child

Reasoning Services

Some reasoning services

- Does a description C make sense at all, i.e., is it satisfiable? A concept description C is satisfiable, if there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.

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  C is subsumed by D (in symbols $C \sqsubseteq D$) if we have for all interpretations $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

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  a is an instance of C if for all interpretations, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

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- Is a an instance of a concept C?
  a is an instance of C if for all interpretations, we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
- *Note*: These questions can be posed with or without a TBox that restricts the possible interpretations.

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Outlook

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- Can we reduce the reasoning services to perhaps just one problem?
- What could be reasoning algorithms?
- What can we say about complexity and decidability?
- What has all that to do with modal logics?
- How can one build efficient systems?

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Literature I

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Summary: Concept descriptions

Abstract	Concrete	Interpretation
A	A	$A^{\mathcal{I}}$
$C \sqcap D$	(and C D)	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$C \sqcup D$	(or C D)	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
$\neg C$	(not C)	$\mathcal{D} - C^{\mathcal{I}}$
$\forall r.C$	(all r C)	$\{d \in \mathcal{D} : r^{\mathcal{I}}(d) \subseteq C^{\mathcal{I}}\}$
$\exists r$	(some r)	$\{d \in \mathcal{D} : r^{\mathcal{I}}(d) \neq \emptyset\}$
$\geq n\ r$	(atleast n r)	$\{d \in \mathcal{D} : \|r^{\mathcal{I}}(d)\| \geq n\}$
$\leq n\ r$	(atmost n r)	$\{d \in \mathcal{D} : \|r^{\mathcal{I}}(d)\| \leq n\}$
$\exists r.C$	(some r C)	$\{d \in \mathcal{D} : r^{\mathcal{I}}(d) \cap C^{\mathcal{I}} \neq \emptyset\}$
$\geq n\ r.C$	(atleast n r C)	$\{d \in \mathcal{D} : \|r^{\mathcal{I}}(d) \cap C^{\mathcal{I}}\| \geq n\}$
$\leq n\ r.C$	(atmost n r C)	$\{d \in \mathcal{D} : \|r^{\mathcal{I}}(d) \cap C^{\mathcal{I}}\| \leq n\}$
$r \doteq s$	(eq r s)	$\{d \in \mathcal{D} : r^{\mathcal{I}}(d) = s^{\mathcal{I}}(d)\}$
$r \neq s$	(neq r s)	$\{d \in \mathcal{D} : r^{\mathcal{I}}(d) \neq s^{\mathcal{I}}(d)\}$
$r \sqsubseteq s$	(subset r s)	$\{d \in \mathcal{D} : r^{\mathcal{I}}(d) \subseteq s^{\mathcal{I}}(d)\}$
$g \doteq h$	(eq g h)	$\{d \in \mathcal{D} : g^{\mathcal{I}}(d) = h^{\mathcal{I}}(d) \neq \emptyset\}$
$g \neq h$	(neq g h)	$\{d \in \mathcal{D} : \emptyset \neq g^{\mathcal{I}}(d) \neq h^{\mathcal{I}}(d) \neq \emptyset\}$
$\{i_1, i_2, \ldots, i_n\}$	(oneof $i_1 \ldots i_n$)	$\{i_1^{\mathcal{I}}, i_2^{\mathcal{I}}, \ldots, i_n^{\mathcal{I}}\}$

Summary: Role descriptions

Abstract	Concrete	Interpretation
t	t	$t^{\mathcal{I}}$
f	f	$f^{\mathcal{I}}$, **(functional role)**
$r \sqcap s$	(and r s)	$r^{\mathcal{I}} \cap s^{\mathcal{I}}$
$r \sqcup s$	(or r s)	$r^{\mathcal{I}} \cup s^{\mathcal{I}}$
$\neg r$	(not r)	$\mathcal{D} \times \mathcal{D} - r^{\mathcal{I}}$
r^{-1}	(inverse r)	$\left\{(d,d') : (d',d) \in r^{\mathcal{I}}\right\}$
$r\vert_C$	(restr r C)	$\left\{(d,d') \in r^{\mathcal{I}} : d' \in C^{\mathcal{I}}\right\}$
r^+	(trans r)	$(r^{\mathcal{I}})^+$
$r \circ s$	(compose r s)	$r^{\mathcal{I}} \circ s^{\mathcal{I}}$
1	self	$\{(d,d) : d \in \mathcal{D}\}$