Principles of Knowledge Representation and Reasoning

Complexity Theory

Bernhard Nebel, Stefan Wölfl, and Felix Lindner

October 28, 2015

1 Motivation

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

Why complexity theory?

- Complexity theory can answer questions on how easy or hard a problem is
- Gives hints on what algorithms could be appropriate, e.g.:
  - algorithms for polynomial-time problems are usually easy to design
  - for NP-complete problems, backtracking and local search work well
- Gives hints on what type of algorithm will (most probably) not work
- Gives hint on what sub-problems might be interesting

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

2 Basic Notions: a Reminder

- Algorithms and Turing machines
- Problems, solutions, and complexity
- Complexity classes P and NP
- Upper and lower bounds
- Polynomial reductions
- NP-completeness

Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

Algorithms and Turing machines

- We use Turing machines as formal models of algorithms
- This is justified, because:
  - we assume that Turing machines can compute all computable functions
  - the resource requirements (in term of time and memory) of a Turing machine are only polynomially worse than other models
- The regular type of Turing machine is the deterministic one: DTM (or simply TM)
- Often, however, we use the notion of nondeterministic TMs: NDTM

Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

Problems, solutions, and complexity

- A problem is a set of pairs (I, A) of strings in $\{0,1\}^*$.
  I: instance; A: answer
  If all answers $A \in \{0,1\}$: decision problem
- A decision problem is the same as a formal language: the set of strings formed by the instances with answer 1
- An algorithm solves (or decides) a problem if it computes the right answer for all instances.
- Complexity of an algorithm: function

$$T : \mathbb{N} \to \mathbb{N},$$

  measuring the number of basic steps (or memory requirement) the algorithm needs to compute an answer depending on the size of the instance
- Complexity of a problem: complexity of the most efficient algorithm that solves this problem.

Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

Complexity classes P and NP

Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: P
  - Problems in P are assumed to be efficiently solvable (although this might not be true if the exponent is very large)
  - In practice, this notion appears to be more often reasonable than not
- The class of problems decidable on non-deterministic Turing machines in polynomial time: NP
- More classes are definable using other resource bounds on time and memory

Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

Upper and lower bounds

- Upper bounds (membership in a class) are usually easy to prove:
  - provide an algorithm
  - show that the resource bounds are respected
- Lower bounds (hardness for a class) are usually difficult to show:
  - the technical tool here is the polynomial reduction (or any other appropriate reduction)
  - show that some hard problem can be reduced to the problem at hand

Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

Polynomial reduction

- Given languages L_1 and L_2, L_1 can be polynomially reduced to L_2, written $L_1 \leq_p L_2$, if there exists a polynomial time-computable function f such that

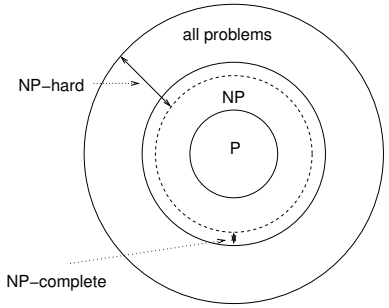
$$x \in L_1 \iff f(x) \in L_2.$$

  Rationale: it cannot be harder to decide L_1 than L_2
- L is hard for a class C (C-hard) if all languages of this class can be reduced to L.
- L is complete for C (C-complete) if L is C-hard and $L \in C$.

Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

NP-complete problems

- A problem is NP-complete iff it is NP-hard and in NP.
- Example: SAT (the satisfiability problem for propositional logic) is NP-complete (Cook/Karp)
  - Membership is obvious, hardness follows because computations on a NDTM correspond to satisfying truth assignments of certain formulae



Motivation
Basic Notions: a Reminder
Algorithms and Turing machines
Problems, solutions, and complexity
Complexity classes P and NP
Upper and lower bounds
Polynomial reductions
NP-completeness
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Literature

3 Beyond NP

- The class co-NP
- The class PSPACE
- Other classes

Motivation
Basic Notions: a Reminder
Beyond NP
The class co-NP
The class PSPACE
Other classes
Oracle TMs and the Polynomial Hierarchy
Literature

The complexity class co-NP

- Note that there is some asymmetry in the definition of NP:
  - It is clear that we can decide SAT by using a NDTM with polynomially bounded computation
  - There exists an accepting computation of polynomial length iff the formula is satisfiable
  - What if we want to solve UNSAT, the complementary problem?
  - It seems necessary to check all possible truth-assignments!
- Define co-C = $\{L \subseteq \Sigma^* : \Sigma^* \setminus L \in C\}$ (provided Σ is our alphabet)
- co-NP = $\{L \subseteq \Sigma^* : \Sigma^* \setminus L \in \mathrm{NP}\}$
- Examples: UNSAT, TAUT $\in$ co-NP!
- *Note:* P is closed under complement, in particular,

$$\mathrm{P} \subseteq \mathrm{NP} \cap \text{co-NP}$$

Motivation
Basic Notions: a Reminder
Beyond NP
The class co-NP
The class PSPACE
Other classes
Oracle TMs and the Polynomial Hierarchy
Literature

PSPACE

There are problems even more difficult than NP and co-NP...

Definition ((N)PSPACE)

PSPACE (NPSPACE) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only polynomially many tape cells.

Some facts about PSPACE:

- PSPACE is closed under complements (... as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- NP$\subseteq$PSPACE (because in polynomial time one can "visit" only polynomial space, i.e., NP$\subseteq$NPSPACE)
- It is unknown whether NP$\neq$PSPACE, but it is believed that

Motivation
Basic Notions: a Reminder
Beyond NP
The class co-NP
The class PSPACE
Other classes
Oracle TMs and the Polynomial Hierarchy
Literature

PSPACE-completeness

Definition (PSPACE-completeness)

A decision problem (or language) is PSPACE-complete if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, PSPACE-complete problems are the "hardest" problems in PSPACE (similar to NP-completeness). They appear to be "harder" than NP-complete problems from a practical point of view.

An example for a PSPACE-complete problem is the NDFA equivalence problem:

Instance: Two non-deterministic finite state automata A_1 and A_2.

Question: Are the languages accepted by A_1 and A_2 identical?

Motivation
Basic Notions: a Reminder
Beyond NP
The class co-NP
The class PSPACE
Other classes
Oracle TMs and the Polynomial Hierarchy
Literature

Other complexity classes ...

- There are complexity classes above PSPACE (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME ...)
- There are (infinitely many) classes between NP and PSPACE (the polynomial hierarchy defined by oracle machines)
- There are (infinitely many) classes inside P (circuit classes with different depths)
- ... and for most of the classes we do not know whether the containment relationships are strict

Motivation
Basic Notions: a Reminder
Beyond NP
The class co-NP
The class PSPACE
Other classes
Oracle TMs and the Polynomial Hierarchy
Literature

4 Oracle TMs and the Polynomial Hierarchy

- Oracle Turing machines
- Turing reduction
- Complexity classes based on OTMs
- QBF

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Oracle Turing machines

- An Oracle Turing machine ((N)OTM) is a Turing machine (DTM, NDTM) with the possibility to query an oracle (i. e., a different Turing machine without resource restrictions) whether it accepts or rejects a given string.
- Computation by the oracle does not cost anything!
- Formalization:
  - a tape onto which strings for the oracle are written,
  - a yes/no answer from the oracle depending on whether it accepts or rejects the input string.
- Usage of OTMs answers what-if questions: What if we could solve the oracle-problem efficiently?

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Turing reductions

- OTMs allow us to define a more general type of reduction
- Idea: The "classical" reduction can be seen as calling a subroutine once.
- L_1 is Turing-reducible to L_2, symbolically $L_1 \leq_T L_2$, if there exists a poly-time OTM that decides L_1 by using an oracle for L_2.
- Polynomial reducibility implies Turing reducibility, but not vice versa!
- NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent!
- Turing reducibility can also be applied to general search problems!

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Complexity classes based on Oracle TMs

1. P^{NP} = decision problems solved by poly-time DTMs with an oracle for a decision problem in NP.
2. NP^{NP} = decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
3. co-NP^{NP} = complements of decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
4. $NP^{NP^{NP}}$ = ...

... and so on

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Example

Consider the Minimum Equivalent Expression (MEE) problem:

Instance: A well-formed Boolean formula φ using the standard connectives (not $\leftrightarrow$) and a non-negative integer k.

Question: Is there a well-formed Boolean formula φ' that contains k or fewer literal occurrences and that is logically equivalent to φ?

- This problem is NP-hard (wrt. to Turing reductions).
- It does not appear to be NP-complete.
- We could guess a formula and then use a SAT-oracle ...
- MEE $\in NP^{NP}$.

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

The polynomial hierarchy

The complexity classes based on OTMs form an infinite hierarchy.

The polynomial hierarchy PH

$$\begin{array}{lcllcllcl} \Sigma_0^p & = & \mathrm{P} & \Pi_0^p & = & \mathrm{P} & \Delta_0^p & = & \mathrm{P} \\ \Sigma_{i+1}^p & = & \mathrm{NP}^{\Sigma_i^p} & \Pi_{i+1}^p & = & \text{co-}\Sigma_{i+1}^p & \Delta_{i+1}^p & = & \mathrm{P}^{\Sigma_i^p} \end{array}$$

- PH = $\bigcup_{i \geq 0} (\Sigma_i^p \cup \Pi_i^p \cup \Delta_i^p) \subseteq$ PSPACE
- NP = Σ_1^p
- co-NP = Π_1^p

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Quantified Boolean formulae: definition

- If φ is a propositional formula, P is the set of Boolean variables used in φ and σ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma\varphi$ is a QBF.
- A formula $\exists x \varphi$ is true if and only if $\varphi[x/\top] \vee \varphi[x/\bot]$ is true (equivalently, $\varphi[x/\top]$ is true or $\varphi[x/\bot]$ is true).
- A formula $\forall x \varphi$ is true if and only if $\varphi[x/\top] \wedge \varphi[x/\bot]$ is true (equivalently, $\varphi[x/\top]$ is true and $\varphi[x/\bot]$ is true).
- This definition directly leads to an AND/OR tree traversal algorithm for evaluating QBF.

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Quantified Boolean formulae: definition

The evaluation problem of QBF generalizes both the satisfiability and validity/tautology problems of propositional logic.
The latter are NP-complete and co-NP-complete, resp., whereas the former is PSPACE-complete.

Example

The formulae $\forall x \exists y (x \leftrightarrow y)$ and $\exists x \exists y (x \wedge y)$ are true.

Example

The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \vee y)$ are false.

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

The Polynomial Hierarchy: connection to QBF

Truth of QBFs with prefix $\overbrace{\forall\exists\forall\ldots}^{i}$ is Π_i^p-complete.

Truth of QBFs with prefix $\overbrace{\exists\forall\exists\ldots}^{i}$ is Σ_i^p-complete.

Special cases corresponding to SAT and TAUT:

- The truth of QBFs with prefix $\exists x_1^1 \ldots x_n^1$ is NP= Σ_1^p-complete.
- The truth of QBFs with prefix $\forall x_1^1 \ldots x_n^1$ is co-NP= Π_1^p-complete.

Motivation
Basic Notions: a Reminder
Beyond NP
Oracle TMs and the Polynomial Hierarchy
Oracle Turing machines
Turing reduction
Complexity classes based on OTMs
QBF
Literature

Literature

Motivation

Basic Notions: a Reminder

Beyond NP

Oracle TMs and the Polynomial Hierarchy

Literature

M. R. Garey and D. S. Johnson.
Computers and Intractability – A Guide to the Theory of NP-Completeness.
Freeman and Company, San Francisco, 1979.

C. H. Papadimitriou.
Computational Complexity.
Addison-Wesley,Reading, MA, 1994.