

Constraint Satisfaction Problems

Qualitative Temporal CSP

Albert-Ludwigs-Universität Freiburg



UNI
FREIBURG

Stefan Wöfl, Christian Becker-Asano, and Bernhard Nebel

February 4 & 9, 2015



Motivation

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Quantitative vs. qualitative representations



Spatio-temporal configurations can be described **quantitatively** by specifying the coordinates of the relevant objects:

Example: At time point 10.0 object *A* is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object *B* is at position (15.2, 3.5, 23.7). Object *C* is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

Example: Object *A* hit object *B*. Afterwards, object *C* arrived.

Sometimes we want to reason with such descriptions.

Example: Object *C* was not close to object *A*, when it hit object *B*.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Quantitative vs. qualitative representations



Spatio-temporal configurations can be described **quantitatively** by specifying the coordinates of the relevant objects:

Example: At time point 10.0 object *A* is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object *B* is at position (15.2, 3.5, 23.7). Object *C* is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).

Often, however, a **qualitative** description (using a finite vocabulary) is more adequate:

Example: Object *A* hit object *B*. Afterwards, object *C* arrived.

Sometimes we want to reason with such descriptions.

Example: Object *C* was not close to object *A*, when it hit object *B*.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Quantitative vs. qualitative representations



Spatio-temporal configurations can be described **quantitatively** by specifying the coordinates of the relevant objects:

Example: At time point 10.0 object *A* is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object *B* is at position (15.2, 3.5, 23.7). Object *C* is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).

Often, however, a **qualitative** description (using a finite vocabulary) is more adequate:

Example: Object *A* hit object *B*. Afterwards, object *C* arrived.

Sometimes we want to reason with such descriptions.

Example: Object *C* was not close to object *A*, when it hit object *B*.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Intention: describe configurations in an infinite (continuous) domain using a finite vocabulary and reason about these descriptions

- Specification of a **vocabulary**: usually a finite set of relations (often binary) that are **pairwise disjoint** and **jointly exhaustive**
- Specification of a **language**: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal **semantics**
- Analysis of computational properties and design of **reasoning methods** (often constraint propagation)
- Perhaps, specification of **operational semantics** for verifying whether a relation holds in a given quantitative configuration

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- ...

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Example: Qualitative temporal relations



Suppose, we want to talk about **time instants** (points) and binary **relations** over them.

- **Vocabulary:** $X = Y$ (X equals Y), $X < Y$ (X before Y), and $X > Y$ (X after Y).
- **Language:**
 - Allow for **disjunctions** of basic relations to express **indefinite information**. Use unions of relations to express that. For instance, $< \cup =$ expresses \leq .
 - 2^3 different relations (including the **impossible** and the **universal** relation)
 - Use **sets of atomic formulae** with these relations to describe **configurations**. For example:

$$\{x = y, y (< \cup >) z\}$$

- **Semantics:** Interpret the time point symbols and relation symbols over the **real** (or rational) numbers.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



$$\{x(< \cup =)y, y(< \cup =)z, v(< \cup =)y, w > y, z(< \cup =)x\}$$

- **Satisfiability**: Are there values for all time points such that all formulae are satisfied?
- **Satisfiability** with $v = w$?
- Finding a satisfying **instantiation** of all time points
- **Deduction**: Does $x\{=\}y$ follow logically?
Does $v \leq w$ follow?
- Finding a **minimal description**: What are the most constrained relations that describe the same set of instantiations?

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in [satisfiability](#), [model finding](#), and [deduction](#)

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Let \mathcal{B} be a finite set of (binary) relations on some (infinite) domain D (elements of \mathcal{B} are called **base relations**).

We require:

- The relations in \mathcal{B} are **JEPD**, i.e., jointly exhaustive and pairwise disjoint.
- \mathcal{B} is **closed under converses**.

Then:

- Let \mathcal{A} be the set of relations that can be built by taking the unions of relations from \mathcal{B} ($\rightsquigarrow 2^{|\mathcal{B}|}$ different relations).
- \mathcal{A} is closed under converse, complement, intersection and union.
- Often, \mathcal{A} is **closed under composition of base relations**, i.e., for all $B, B' \in \mathcal{B}$,

$$B \circ B' \in \mathcal{A}.$$

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Computing operations on relations



Let \mathcal{A} be the system of relations over a set of **base relations** \mathcal{B} that satisfies all the conditions above.

We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \cong \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \circ B'_j$$

Converse:

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and **union** are defined in the usual set-theoretical way.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Computing operations on relations



Let \mathcal{A} be the system of relations over a set of **base relations** \mathcal{B} that satisfies all the conditions above.

We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \cong \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \circ B'_j$$

Converse:

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and **union** are defined in the usual set-theoretical way.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Computing operations on relations



Let \mathcal{A} be the system of relations over a set of **base relations** \mathcal{B} that satisfies all the conditions above.

We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \cong \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \circ B'_j$$

Converse:

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and union are defined in the usual set-theoretical way.

Computing operations on relations



Let \mathcal{A} be the system of relations over a set of **base relations** \mathcal{B} that satisfies all the conditions above.

We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \cong \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \circ B'_j$$

Converse:

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and **union** are defined in the usual set-theoretical way.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition xRy : Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y of variables the strongest constrained (minimal) relation entailed by the CSP.

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.

$CSAT \leq_T CENT$ and $CENT \leq_T CMIN$ are obvious.

$CENT \leq_T CSAT$: We solve CENT ($CSP \models xRy?$) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let B_1, \dots, B_k be the relations for which we get a positive answer. Then $x\{B_1, \dots, B_k\}y$ is entailed by the CSP.

$CMIN \leq_T CENT$: We use entailment for computing the minimal constraint for each pair of variables. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

Motivation
Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Given a qualitative CSP with $R_{V_1, V_2} = R_{V_2, V_1}^{-1}$. Then the **path consistency method** is to apply the operation

$$R_{V_1, V_2} \leftarrow R_{V_1, V_2} \cap (R_{V_1, V_3} \circ R_{V_3, V_2}).$$

on all the constraints of the network until a fixpoint is reached.

The path consistency method **guarantees** ...

- sometimes **minimality**
- sometimes **satisfiability**
- however sometimes the CSP is **not satisfiable**, even if the CSP contains only **base relations**

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Example: Point relations



Composition table:

	<	=	>
<	<	<	<, =, >
=	<	=	>
>	<, =, >	>	>

Abbildung: Composition table for the point algebra. For example:

$$\{\langle\} \circ \{=\} = \{\langle\}$$

- $\{\langle, =\} \circ \{\langle\} = \{\langle\}$
- $\{\langle, \>\} \circ \{\langle\} = \{\langle, =, \>\}$
- $\{\langle, =\}^{-1} = \{\>, =\}$
- $\{\langle, =\} \cap \{\>, =\} = \{=\}$

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature

Some properties of the point relations



Theorem

A path consistent CSP over the point relations is satisfiable.

In particular, the path consistency method decides satisfiability.

Theorem

A path consistent CSP over all point relations without $\{<, >\}$ is minimal.

Proofs later ...

Motivation

Qualitative CSP

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Qualitative Constraint Languages

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature



From now on, let D be a finite or infinite domain.

Definition

A **partition scheme** on D is any non-empty, finite set Δ of binary relations on D such that:

- Δ defines a partition of $D \times D$.
- Δ contains the binary identity relation id_D .
- Δ is closed under converses.

Definition

A constraint language of binary relations on D , Γ , is said to be **generated** from a partition scheme Δ , if Γ consists of all finite unions of relations in Δ .

Constraint languages in this sense will be referred to as **qualitative constraint languages**.

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature



Let Γ be a subset of a qualitative constraint language with partition scheme Δ .

Definition

A **qualitative constraint network** over Γ is a triple

$$P = \langle V, D, C \rangle,$$

where:

- V is a non-empty and finite set of **variables**,
- D is an arbitrary non-empty set (**domain**),
- C is a finite set of **constraints** C_1, \dots, C_q , i.e., each constraint C_i is a pair (s_i, R_i) , where s_i is a pair of variables and R_i is a binary relation contained in Γ .

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature

Let Γ be a qualitative constraint language with partition scheme Δ . For $R, S \in \Gamma$, define:

$$R \circ_w S := \bigcup \{T \in \Delta : T \cap (R \circ S) \neq \emptyset\}$$

\circ_w is called **weak composition** of R and S .

Lemma

For all relations $R, S, T \in \Gamma$,

- $R \circ S \subseteq R \circ_w S$;
- $T \cap (R \circ S) = \emptyset$ if and only if $T \cap (R \circ_w S) = \emptyset$;
- $(R \circ_w S)^{-1} = S^{-1} \circ_w R^{-1}$;
- $R \circ_w (S \cup T) = (R \circ_w S) \cup (R \circ_w T)$.

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature

Example:

Consider a linear order on a domain with 2 elements $a < b$. The relations $R_<, R_=: R_>$ define a partition schema on D . It holds:

$$R_< \circ R_< = R_> \circ R_> = \emptyset, \quad R_< \circ R_> = \{(a, a)\}, \quad R_> \circ R_< = \{(b, b)\}$$

but

$$R_< \circ_w R_< = R_> \circ_w R_> = \emptyset, \quad R_< \circ_w R_> = R_=: R_> \circ_w R_< = R_=:$$

Moreover,

$$(R_< \circ_w R_>) \circ_w R_> = R_=: \circ_w R_> = R_> \neq \emptyset = R_< \circ_w \emptyset = R_< \circ_w (R_> \circ_w R_>).$$

Example:

Consider a linear order on a domain with 3 elements $a < b < c$. Then

$$R_< \circ R_< = \{(a, c)\} \quad \text{but} \quad R_< \circ_w R_< = R_<.$$

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature



Let Γ be a qualitative constraint language with partition scheme Δ . As spelled out before, each relation R in Γ can be represented by a finite disjunction of “base relations” $B_1, \dots, B_k \in \Delta$. In what follows we identify R with the set of its base relations

$$\{B_1, \dots, B_k\}.$$

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature

Lemma

For each partition scheme Δ , the tuple

$$\langle 2^\Delta, \cap, \cup, \circ_w, \mathbf{C}_\Delta, {}^{-1}, \emptyset, \Delta, \text{id}_\Delta \rangle$$

defines a non-associative relation algebra.

A qualitative network $P = \langle V, D, C \rangle$ is **normalized**, if

- for each pair of variables x, y , C contains at most one constraint $((x, y), R)$ and one constraint $((y, x), S)$;
- for each constraint $((x, x), R)$ in C , $R = \text{id}_D$;
- for constraints $((x, y), R)$ and $((y, x), S)$ in C , $R = S^{-1}$.

In what follows we will always assume that constraint networks are normalized.

Definition

A qualitative constraint network P is **algebraically closed** (or: **a-closed**), if for all constraints $((x, y), R)$, $((x, z), S)$, and $((z, y), T)$ of P , it holds:

$$R \subseteq S \circ_w T.$$

Note: If P is algebraically closed, then $R = R \cap (S \circ_w T)$.

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature



The **path consistency algorithm** can only be used if the underlying partition scheme is closed under composition, i.e., if for each pair of relations $R, S \in \Delta$, $R \circ S$ is a (finite) union of a subset of Δ .

The **algebraic closure algorithm** is a variant of the path consistency algorithm. Instead of ordinary composition of relations, we use weak composition.

Since weak composition is an upper approximation of composition only, the algebraic closure algorithm may not result in a path-consistent network.

Let $P = \langle V, D, C \rangle$ be a (normalized) qualitative constraint network.

Let $Table[i, j]$ be a $n \times n$ -matrix (n : number of variables), in which we record the constraints between the variables.

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature

Algebraic closure algorithm



EnforceAlgClosure (P):

Input: a qualitative network $P = \langle V, D, C \rangle$

Output: “inconsistent”, or an equivalent algebraically closed network P'

$$\text{Paths}(i,j) = \{(i,j,k) : 1 \leq k \leq n, k \neq i,j\} \cup \\ \{(k,i,j) : 1 \leq k \leq n, k \neq i,j\}$$

$\text{Queue} := \bigcup_{i,j} \text{Paths}(i,j)$

while $Q \neq \emptyset$

 select and delete (i,k,j) from Q

$T := \text{Table}[i,j] \cap (\text{Table}[i,k] \circ_w \text{Table}[k,j])$

if $T = \emptyset$

return “inconsistent”

elseif $T \neq \text{Table}[i,j]$

$\text{Table}[i,j] := T$

$\text{Table}[j,i] := T^{-1}$

$\text{Queue} := \text{Queue} \cup \text{Paths}(i,j)$

return P' with the refined constraints as recorded in Table

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature



Let Γ be a qualitative constraint language with partition scheme Δ .

We suppose that we have determined (by some formal proof or some computation) the (weak) composition table for Δ , i.e.,

$$\circ_{(w)} : \Delta \times \Delta \rightarrow 2^{\Delta}.$$

Let now B be a finite set of symbols (bijective with Δ). Then 2^B is a Boolean algebra, from which we obtain a (non-associative) relation algebra, if we extend $\circ_{(w)}$ to a function

$$\circ_{(w)} : 2^B \times 2^B \rightarrow 2^B.$$

Now we can perform all the operations needed in the path consistency/a-closure algorithm on the symbolic level.

Motivation

Qualitative
Constraint
Languages

Constraint
Propagation

Allen's
Interval
Algebra

Literature



Allen's Interval Algebra

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



- Allen's interval calculus (IA): **time intervals** and **binary relations** over them
- Let $\langle \mathbb{R}, < \rangle$ be the linear order on the real numbers (conceived of as the **flow of time**).

Then, the **domain** D of Allen's calculus is the set of all *intervals*

$$X = (X^-, X^+) \in \mathbb{R}^2, \text{ where } X^- < X^+$$

(naïve approach)

- **Relations** between concrete intervals, e. g.:

(1.0,2.0) strictly before (3.0,5.5)

(1.0,3.0) meets (3.0,5.5)

(1.0,4.0) overlaps (3.0,5.5)

...

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

To determine all possible relation between Allen intervals, we determine how one can order the four points of two intervals:

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	\prec	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	m	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	o	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	s	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	f	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	d	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	\equiv	equal

and the *converse* relations (obtained by exchanging X and Y)

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

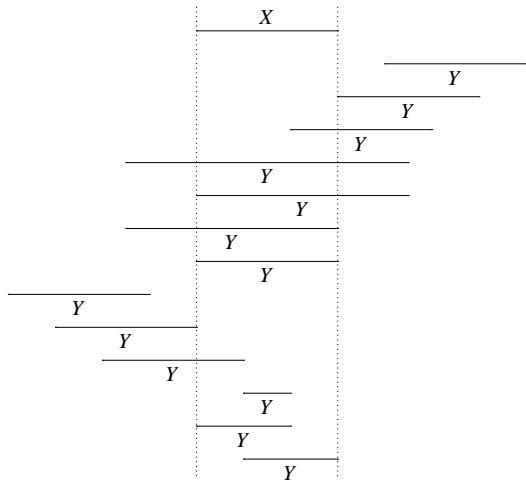
The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: The 13 base relations graphically



before⁻¹
meets⁻¹
overlaps⁻¹
during⁻¹
starts⁻¹
finishes⁻¹
equals⁻¹
before⁻¹
meets⁻¹
overlaps⁻¹
during⁻¹
starts⁻¹
finishes⁻¹

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Lemma

*The 13 base relations of Allen's interval calculus define a **partition scheme** on the set of all Allen intervals.*

In what follows:

- **IA**: the qualitative constraint language generated from all base relations of Allen's interval calculus (contains $2^{13} = 8192$ relations)
- **IA- \mathcal{B}** : the subclass of IA containing base relations only

Lemma

The set of base relations of Allen's interval calculus is closed under composition.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

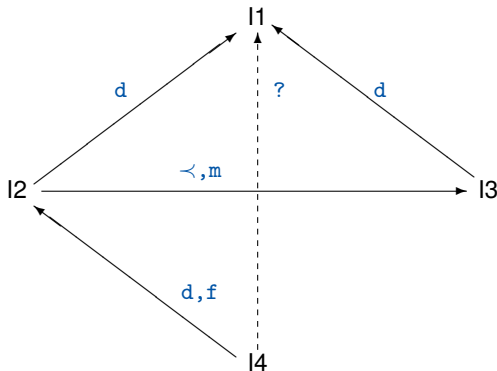
The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: An example



Compose the constraints: $I4 \{d, f\} I2$ and $I2 \{d\} I1$: $I4 \{d\} I1$.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

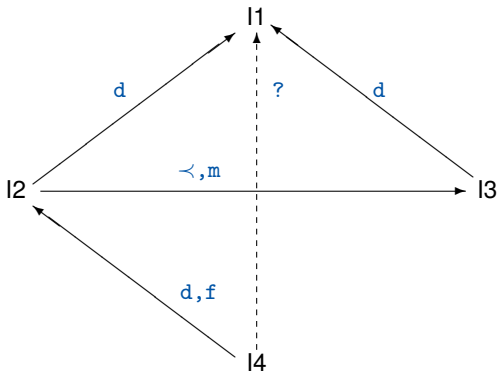
The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: An example



Compose the constraints: $I4 \{d, f\} I2$ and $I2 \{d\} I1$: $I4 \{d\} I1$.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

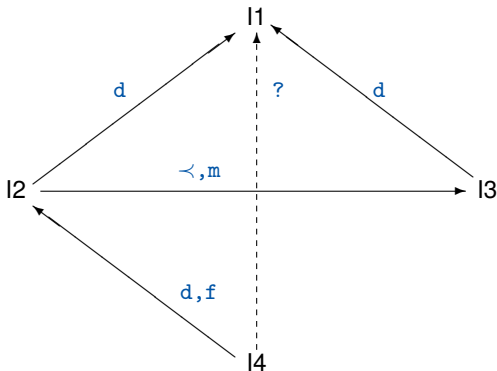
The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Compose the constraints: $I4 \{d, f\} I2$ and $I2 \{d\} I1$: $I4 \{d\} I1$.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

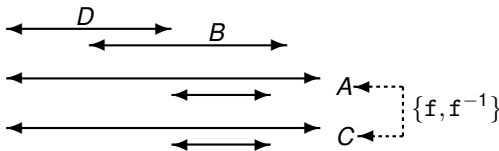
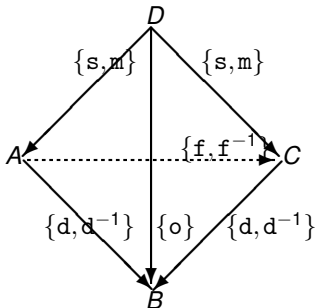
The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: Example for incompleteness



Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

**IA: Example for
Incompleteness**

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

Theorem (Kautz & Vilain)

Deciding satisfiability over IA is NP-hard.

Proof.

Reduction from **3-colorability** (the original proof uses 3Sat).

Let $G = (V, E)$, $V = \{v_1, \dots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \dots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{llll} 1 & \{m\} & 2 & \\ 2 & \{m\} & 3 & \\ v_i & \{m, \equiv, m^{-1}\} & 2 & \forall v_i \in V \\ v_i & \{m, m^{-1}, <, >\} & v_j & \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable **iff** G can be colored with 3 colors. □

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

**IA: Example for
Incompleteness**

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Following, we will look at polynomial special cases, i.e., subclasses of the qualitative constraint language IA.

For this we start from a natural translation of interval relations/constraints (of the form $X R Y$) into **clause formulas** over **atoms** of the form $a op b$, where:

- $a, b \in \{X^-, X^+, Y^-, Y^+\}$; and
- $op \in \{<, >, =, \leq, \geq\}$.

Example: All base relations can be expressed as unit clauses.

Lemma

Let P be a constraint network over IA, and let $\pi(P)$ be the translation of P into clause form.

P is satisfiable iff $\pi(P)$ is satisfiable over the real numbers.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

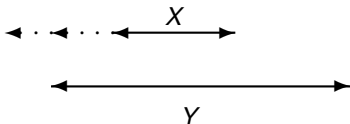
IA: The Continuous Endpoint Class



Continuous Endpoint Class IA- \mathcal{C} : the subset of IA consisting of those relations with a clause form containing only unit clauses, where $\neg(a = b)$ is **forbidden**.

Example: All basic relations and, e.g., $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \left\{ \begin{array}{l} X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+ \end{array} \right\}$$



The set IA- \mathcal{C} contains 83 relations. It is **closed** under intersection, composition, and converses (it is a **sub-algebra** wrt. these three operations on relations). This can be shown by using

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

**The Continuous
Endpoint Class**

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



We will prove:

Lemma

Each 3-consistent interval CSP over IA- \mathcal{C} is globally consistent.

From this we can conclude:

Theorem (van Beek)

Applied to networks over IA- \mathcal{C} , enforcing path consistency decides satisfiability and solves the minimal label problem.

Corollary

A path-consistent interval constraint network containing base relations only is satisfiable.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

**The Continuous
Endpoint Class**

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Definition

A set $M \subseteq \mathbb{R}^n$ is **convex** iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M .

Theorem (Helly)

Let F be a family of at least $n + 1$ convex sets in \mathbb{R}^n . If all sub-families of F with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

**The Continuous
Endpoint Class**

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: Strong n -consistency (1)



Proof of the lemma.

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3$ ✓

Induction assumption: Assume strong $k - 1$ -consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables X_i to pairs $(s_i, e_i) \in \mathbb{R}^2$ satisfying the constraints R_{ij} between the $k - 1$ variables.

We have to show that we can extend the instantiation to any k th variable.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

**The Continuous
Endpoint Class**

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: Strong n -consistency (2): Instantiating the k th Variable



Proof (Part 2).

The instantiation of the $k - 1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k .

Note: Since $R_{ij} \in \text{IA-}\mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in \mathbb{R}^2 .

\rightsquigarrow Consider sets of 3 inequalities (= 3 convex sets).

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

**The Continuous
Endpoint Class**

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: Strong n -consistency (3) – using Helly's theorem



Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+ , but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by the same argument as above there exists a common point.

- ↪ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.
- ↪ Strong k -consistency for all $k \leq n$.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

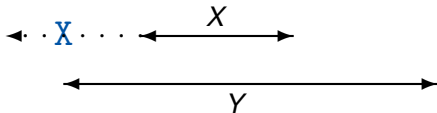
IA: The Endpoint Subclass



Endpoint Subclass: IA- \mathcal{P} is the subclass that permits a clause form containing only **unit** clauses ($a \neq b$ is now allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \{ X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+ \}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path consistency method decides satisfiability over IA- \mathcal{P} .

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

**The Endpoint
Subclass**

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: The ORD-Horn Subclass



ORD-Horn Subclass: IA- \mathcal{H} is the subclass of IA that permits a clause form containing only **Horn clauses**, where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \text{IA-}\mathcal{P}$ and $\{o, s, f^{-1}\}$:

$$\pi(X\{o, s, f^{-1}\}Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

**The ORD-Horn
Subclass**

Solving Arbitrary
Allen CSPs

Outlook

Literature



Let *ORD* be the following theory:

$$\forall x, y, z: \quad x \leq y \wedge y \leq z \quad \rightarrow \quad x \leq z \quad (\textit{transitivity})$$

$$\forall x: \quad x \leq x \quad (\textit{reflexivity})$$

$$\forall x, y: \quad x \leq y \wedge y \leq x \quad \rightarrow \quad x = y \quad (\textit{anti-symmetry})$$

$$\forall x, y: \quad x = y \quad \rightarrow \quad x \leq y \quad (\textit{weakening of =})$$

$$\forall x, y: \quad x = y \quad \rightarrow \quad y \leq x \quad (\textit{weakening of =}).$$

- *ORD* describes partially ordered sets, \leq being the ordering relation.
- *ORD* is a [Horn theory](#)
- What is missing wrt. *dense* and *linear* orders?

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The *ORD*-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Lemma

Let Θ be a CSP over IA- \mathcal{H} . Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

\Rightarrow : Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.

\Leftarrow : Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

**The ORD-Horn
Subclass**

Solving Arbitrary
Allen CSPs

Outlook

Literature



Lemma

Let Θ be a CSP over IA- \mathcal{H} . Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

\Rightarrow : Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.

\Leftarrow : Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature

Complexity of CSAT($IA-\mathcal{H}$)



Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Lemma

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Theorem

$CSAT(IA-\mathcal{H})$ can be decided in polynomial time.

Proof.

$CSAT(IA-\mathcal{H})$ instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. \square

$IA-\mathcal{C} \subset IA-\mathcal{P} \subset IA-\mathcal{H}$ with $|IA-\mathcal{C}| = 83$, $|IA-\mathcal{P}| = 188$, $|IA-\mathcal{H}| = 868$

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

**The ORD-Horn
Subclass**

Solving Arbitrary
Allen CSPs

Outlook

Literature

Complexity of CSAT(IA- \mathcal{H})



Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Lemma

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Theorem

CSAT(IA- \mathcal{H}) can be decided in polynomial time.

Proof.

CSAT(IA- \mathcal{H}) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. \square

$IA-\mathcal{C} \subset IA-\mathcal{P} \subset IA-\mathcal{H}$ with $|IA-\mathcal{C}| = 83$, $|IA-\mathcal{P}| = 188$, $|IA-\mathcal{H}| = 868$

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Lemma

Let Θ be a path-consistent set over $IA-\mathcal{H}$. Then

$$(X \{ \} Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides $CSAT(IA-\mathcal{H})$.

- ↪ Maximality of $IA-\mathcal{H}$?
- ↪ Do we have to check all 8192 – 868 extensions?

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

**The ORD-Horn
Subclass**

Solving Arbitrary
Allen CSPs

Outlook

Literature

IA: The ORD-Horn subclass: Maximality



A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras containing all base relations that strictly contain IA- \mathcal{H} : $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause forms of these relations contain “proper” disjunctions!

Theorem

The satisfiability problem over IA- $\mathcal{H} \cup \{N_i\}$ is NP-complete.

Lemma

*IA- \mathcal{H} is the **only** maximal tractable subclass that contains all base relations of IA.*

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

**The ORD-Horn
Subclass**

Solving Arbitrary
Allen CSPs

Outlook

Literature



- Backtracking algorithm using **path consistency** as a **forward-checking method**
 - Method works on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
 - Refinements and evaluation of different heuristics
- ~> Which tractable fragment should one use?

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

**Solving Arbitrary
Allen CSPs**

Outlook

Literature



- If the labels are split into **base relations**, then on average a label is split into

6.5 relations

- If the labels are split into **pointizable relations** (\mathcal{P}), then on average a label is split into

2.955 relations

- If the labels are split into **ORD-Horn relations** (\mathcal{H}), then on average a label is split into

2.533 relations

- ↪ A difference of **0.422** which becomes significant, when applied to extremely hard instances

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the **continuous endpoint class**, minimal CSPs can be computed using the path consistency method.
- For the larger **ORD-Horn class**, CSAT is still decided by the path consistency method.
- Can be used in practice for backtracking algorithms.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



- **Qualitative representation and reasoning** usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- **Reasoning problems** are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with **CSP methods** (but note: **infinite domains**).
- **Path consistency** is the basic reasoning step ... sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Intervals and
Relations Between
Them

IA: Examples

IA: Example for
Incompleteness

The Continuous
Endpoint Class

The Continuous
Endpoint Class

The Endpoint
Subclass

The ORD-Horn
Subclass

Solving Arbitrary
Allen CSPs

Outlook

Literature



Literature

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



J. F. Allen.

Maintaining knowledge about temporal intervals.

Communications of the ACM, 26(11):832–843, November 1983.

Also in **Readings in Knowledge Representation**.



P. van Beek and R. Cohen.

Exact and approximate reasoning about temporal relations.

Computational Intelligence, 6:132–144, 1990.



Peter B. Ladkin and Roger Maddux.

On binary constraint networks.

Journal of the ACM, 41:435–469, 1994.



Alan K. Mackworth.

Constraint satisfaction.

In S. C. Shapiro, editor, **Encyclopedia of Artificial Intelligence**, pages 205–211. Wiley, Chichester, England, 1987.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



Alan K. Mackworth.

Consistency in networks of relations.

Artificial Intelligence, 8:99–118, 1977.



Ugo Montanari.

Networks of constraints: fundamental properties and applications to picture processing.

Information Science, 7:95–132, 1974.



B. Nebel and H.-J. Bürckert.

Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra,

Journal of the ACM, 42(1): 43-66, 1995.



B. Nebel.

Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-horn class.

CONSTRAINTS, 1(3): 175

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature



R. Hirsch.

Tractable approximations for temporal constraint handling.

Artificial Intelligence, 116: 287-295, 2000.

(Contains the pathological set of relations.)



A. Krokhin, P. Jeavons and P. Jonsson.

A complete classification of complexity in Allen's algebra in the presence of a non-trivial basic relation.

Proc. 17th Int. Joint Conf. on AI (IJCAI-01), 83-88, Seattle, WA, 2001.



J. Renz & B. Nebel,

Efficient Methods for Qualitative Spatial Reasoning,

Proceedings of the 13th European Conference on Artificial

Intelligence (ECAI'98), August 1998, 562-566.

Motivation

Qualitative
Constraint
Languages

Allen's
Interval
Algebra

Literature