Constraint Satisfaction Problems

Tractable Constraint Languages

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Expressiveness vs. complexity

- For some restricted constraint languages we know some polynomial time algorithms that solve each instance of that language
- Restricting constraint languages entails restricting expressiveness, i.e., the class of problems that can be expressed in the language

$\rightsquigarrow$ Expressiveness vs computational complexity?

CSP instances aka constraint networks

Definition

An instance of a constraint satisfaction problem (i.e., a constraint network) is a triple

$$N = \langle V, D, C \rangle,$$

where:

- V is a non-empty and finite set of variables,
- D is an arbitrary set (domain),
- C is a finite set of constraints $C_1, \ldots, C_q$, i.e., each constraint C_i is a pair (s_i, R_i), where s_i is a tuple of variables of length m_i and R_i is an m_i-ary relation on D (s_i: constraint scope; R_i: constraint relation).

Restricting the general CSP

The general CSP decision problem is the following: Given an instance of a constraint satisfaction problem, N, determine if there exists solution to N, i.e., determine whether

$$\begin{aligned}&\mathrm{Sol}(N)\\&\qquad := \{(d_1, \ldots, d_n) \in D^n : a(v_i) = d_i \text{ for a solution } a \text{ of } N\}\end{aligned}$$

(where n is the number of variables of V) is not empty.

Restricting the general CSP:

- structural restriction: consider just CSP instances with particular constraint scopes (e. g., where the network hypergraph has specific properties)
- relational restriction: consider just CSP instances, where the constraint relations have a specific form or specific properties

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Tractable Constraint Languages

Constraint language

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Definition

A constraint language is an arbitrary set of relations, Γ, defined over some fixed domain (denoted by $D(\Gamma)$).

Definition

For a constraint language Γ, let CSP(Γ) be the class of CSP instances $N = \langle V, D, C \rangle$ such that for each $(s, R) \in C$, $R \in \Gamma$. CSP(Γ) is called the relational subclass associated with Γ.

Definition

A finite constraint language Γ is tractable if there exists a polynomial algorithm that solves all instances of CSP(Γ).
An infinite constraint language Γ is tractable if each finite subset of the language is tractable.

Following, we present some examples:

Example: CHIP language

CHIP is a constraint language for arithmetic and other constraints. Basic constraints in CHIP are so-called:

- domain constraints: unary constraints that restrict the domains of variables to a finite set of natural numbers
- arithmetic constraints: constraints of one of the forms

  $$ax = by + c$$
  $$ax \leq by + c$$
  $$ax \geq by + c$$

  ($a, b, c \in \mathbb{N}, a \neq 0$). If these equations are conceived of as relations, the resulting constraint language is tractable.

The language is still tractable if we allow for relations expressed by

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq by + c$$
$$ax_1 \cdots x_n \geq by + c$$
$$(a_1x_1 \geq b_1) \vee \cdots \vee (a_nx_n \geq b_n) \vee (ay \geq b)$$

Example: Linear relations

Let D be any field (e.g., the field of real numbers).
A linear relation on D is any relation defined by some system of linear equations:

$$a_1x_1 + \cdots + a_nx_n = r \qquad (a_1, \ldots, a_n, r \in D).$$

Then any instance of CSP(Γ_{lin}) can be represented by a system of linear equations over D, and hence can be solved in polynomial time (apply Gaussian elimination).

Hence, the language of all linear relations over D is tractable.

Example: Relations on finite orderings

Let D be a finite ordered set.
Consider the binary disequality relation

$$\neq_D = \left\{(d_1, d_2) \in D^2 : d_1 \neq d_2\right\}$$

The class of CSP instances CSP($\{\neq_D\}$) corresponds to the graph colorability problem with $|D|$ colors.
CSP($\{\neq_D\}$) is tractable if $|D| \leq 2$ or $|D| = \infty$, and intractable, otherwise.

The ternary betweenness relation over D is defined by:

$$B_D = \left\{(a, b, c) \in D^3 : a < b < c \vee c < b < a\right\}$$

CSP($\{B_D\}$) is tractable if $|D| \leq 4$, and intractable if $|D| \geq 5$.

Example: Connected row-convex relations

Let $D = \{d_1, \ldots, d_n\}$ be a finite (totally) ordered set.
For a binary relation R over D, the matrix representation of R is an $n \times n$ 0,1-matrix M_R, where $M_R[d, d'] = 1$ iff $(d, d') \in R$.
The pruned matrix representation of R results from the matrix representation of R, when we remove all rows and columns in which only 0's occur.
R is connected row-convex, if in the pruned matrix representation of R, the pattern of 1's is connected along each column, along each row, and forms a connected 2-dimensional region.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The constraint language on any class of connected row-convex relations is tractable.

Example: Boolean constraints

Let $D = \{0, 1\}$.
The class of CSP instances CSP($\{N_D\}$), where

$$N_D = D^3 \setminus \{(0,0,0),(1,1,1)\}$$

is the not-all-equal relation over D, is intractable.
CSP($\{N_D\}$) corresponds to the not-all-equal satisfiability problem (NAE-3SAT), which is known to be NP-hard.

The class of CSP instances CSP($\{T_D\}$), where

$$T_D = \{(0,0,1),(0,1,0),(1,0,0)\},$$

is intractable.
CSP($\{T_D\}$) corresponds to the one-in-three satisfiability problem (1-in-3 SAT).

Example: 0/1/all-relations

Let D be an arbitrary finite set. A relation R over D is called 0/1/all-relation if one of the following conditions holds:

- R is unary;
- $R = D_1 \times D_2$ for subsets D_1, D_2 of D;
- $R = \{(d, \pi(d)) : d \in D_1\}$, for some subset $D_1 \subseteq D$ and some permutation π of D;
- $R = \{(a,b) \in D_1 \times D_2 : a = d_1 \vee b = d_2\}$, for some subsets D_1, D_2 of D and some elements $d_1 \in D_1, d_2 \in D_2$.

The language defined by all 0/1/all-relations is tractable.

It is even maximal tractable: if we add any binary relation over D that is not a 0/1/all-relation, then the resulting constraint language becomes intractable.

max-closed relations

Let $(D, <)$ be a linear order. Define $\max : D \times D \to D$ in the usual way, i.e., $\max(a, b) = a$ if $a > b$, and $\max(a, b) = b$, otherwise. We extend max to a function that can be applied to tuples, i.e., we define $\max : D^k \times D^k \to D^k$ by

$$\max((a_1, \ldots, a_k), (b_1, \ldots, b_k)) := (\max(a_1, b_1), \ldots, \max(a_k, b_k)).$$

Definition

An n-ary relation R over D is max-closed if for all $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in R$,

$$\max((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in R.$$

max-closed relations and tractability

Lemma

Let Γ be a constraint language with max-closed relations only. Then CSP(Γ) is tractable.

Proof:

Enforce generalized arc consistency. If any domain of the resulting network is empty, the network is inconsistent. Otherwise, set each variable to its maximal value, ... $\square$

Example: max-closed relations

Consider the CHIP language. All relations of CHIP are max-closed. Hence any set of equations can be solved by establishing gen. arc consistency.

For example, consider a CSP instance with domain $\{1,\ldots,5\}$, variables $\{v,w,x,y,z\}$, and equations

$$w \neq 3,\ z \neq 5,\ 3v \leq z,\ y \geq z+2,\ 3x+y+z \geq 5w+1,\ wz \geq 2y.$$

Enforcing gen. arc consistency results in:

$$D(v) = \{1\},\ D(w) = \{4\},\ D(x) = \{4,5\},\ D(y) = \{5\},\ D(z) = \{3\}.$$

Hence

$$v \mapsto 1, w \mapsto 4, x \mapsto 5, y \mapsto 5, z \mapsto 3$$

is a solution of the constraint network.

Schaefer's Dichotomy Theorem

Boolean constraint languages

The key result in the literature on tractable constraint languages is Schaefer's Dichotomy Theorem (1978).

Definition

A Boolean constraint language is a constraint language over the two-element domain $D = \{0, 1\}$.

Schaefer's theorem states that any Boolean constraint language is either tractable or NP-complete. Moreover, it provides a classification of all tractable constraint languages.

Schaefer's theorem

Theorem (Schaefer 1978)

Let Γ be a Boolean constraint language. Then Γ is tractable if at least one of the following conditions is satisfied:

1. *Each relation in Γ contains the tuple* $(0, \ldots, 0)$.
2. *Each relation in Γ contains the tuple* $(1, \ldots, 1)$.
3. *Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one negative literal.*
4. *Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one positive literal.*
5. *Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most two literals.*
6. *Each relation in Γ is the set of solutions of a system of linear equations over the finite field with 2 elements.*

Algorithm selector

Let Γ be a Boolean constraint language.

Class 1: any CSP instance N can be solved by simply assigning 0 to each variable of N.

Class 2: cf. Class 1 ($v \mapsto 1$).

Class 6: any CSP instance N can be solved by applying the Gaussian elimination procedure.

Class 5: any CSP instance N can be solved by resolution: in this case CSP(Γ) corresponds to the 2-SAT satisfiability problem and this can be solved efficiently by resolution.

Class 4: any CSP instance N can be solved by unit resolution: here CSP(Γ) corresponds to the Horn-SAT satisfiability problem, which can be solved efficiently by unit resolution.

Class 3: cf. Class 4 (“anti-Horn”).

Relational Clones

Gadgets

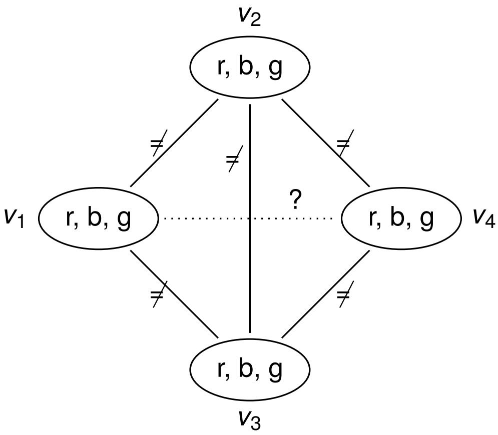
Definition

Let Γ be constraint language and R be a relation on $D(\Gamma)$.
R is expressible in Γ if there exists a CSP instance $N \in \text{CSP}(\Gamma)$ and a sequence of variables $x_1, \ldots, x_r$ in N such that

$$R = \pi_{x_1,\ldots,x_r}(\text{Sol}(N)).$$

N is referred to as a gadget for expressing R in CSP(Γ), the sequence $x_1, \ldots, x_r$ as construction site for R.

Example



Which relation is expressed by the edge (v_1, v_4)?

Relational clones

Expressiveness can also be reformulated in the following way: Let Γ, Γ' be constraint languages (def. on the same domain D).

Definition

Γ' is a relational clone of Γ if Γ' contains each relation definable by a FO-formula with

- relations from $\Gamma \cup \{=_D\}$,
- conjunctions, and
- existential quantification.

(Formulae of this form are called primitive positive formulae.)

Definition

Let Γ be a constraint language. $\langle\Gamma\rangle$ denotes the smallest relational clone containing Γ, the clone generated by Γ.

Example

Consider a Boolean constraint language with the following relations:

$$R_1 = \{(0,1),(1,0),(1,1)\} \qquad R_2 = \{(0,0),(0,1),(1,0)\}.$$

The relational clone generated by the set of these two relations contains all 16 binary Boolean relations. For example:

$$R_3 := \{(0,1),(1,0)\} \qquad R_1(v_1,v_2) \wedge R_2(v_1,v_2)$$

$$R_4 := \{(0,0),(1,0),(1,1)\} \qquad \exists y(R_1(v_1,y) \wedge R_2(y,v_2))$$

$$R_5 := \{(0,0),(1,1)\} \qquad v_1 = v_2$$

$$R_6 := \{(0,0)\} \qquad R_2(v_1,v_2) \wedge R_5(v_1,v_2)$$

$$R_7 := \{(1,1)\} \qquad R_1(v_1,v_2) \wedge R_5(v_1,v_2)$$

$$R_8 := \{(0,1)\} \qquad \exists y(R_6(v_1,y) \wedge R_1(y,v_2))$$

$\dots$

Reducibility I

Theorem

Let Γ be a set of relations on a fixed domain D, and let Δ be a finite subset of $\langle\Gamma\rangle$. Then there exists a polynomial time reduction from $CSP(\Delta)$ to $CSP(\Gamma)$.

Reducibility II

Proof:

Let $\Delta = \{S_1, \ldots, S_k\}$ be a finite set of relations, where each S_j is definable by a pp-formula with relations from Γ and the relation $=_D$. For each S_j fix such a formula $\varphi_j(x_1, \ldots, x_{r_j})$, where r_j is the arity of S_j. Without loss of generality, we may assume that each $\varphi_j(x_1, \ldots, x_{r_j})$ has the form

$$\exists u_1 \ldots u_m (R_1(w_1^1, \ldots, w_{k_1}^1) \wedge \cdots \wedge R_n(w_1^n, \ldots, w_{k_n}^n)) \quad (1)$$

where $w_1^1, \ldots, w_{k_1}^1, \ldots, w_1^n, \ldots, w_{k_n}^n \in \{x_1, \ldots, x_{r_j}, u_1, \ldots, u_m\}$ for some auxiliary variables $u_1, \ldots, u_m$, and $R_1, \ldots, R_n \in \Gamma \cup \{=_D\}$. ...

Reducibility III

...

Let $N = \langle V, D, C \rangle$ be an arbitrary instance in CSP(Δ). Initially, set $V' := V, D' := D, C' := C$. For each constraint (s, R) (where $s = (v_1, \dots, v_r)$) of N, proceed as follows:

1. add the auxiliary variables $u_1, \dots, u_m$ to V' (always add new variables, rename variables if necessary (also in (1)))
2. remove (r, R) from C' and instead add to C' the constraints (cf. (1)):
$$((w^1_1, \dots, w^1_{k_1}), R_1), \dots, (w^n_1, \dots, w^n_{k_n}, R_n)$$

The CSP instance N' obtained by this procedure is contained in CSP($\Gamma \cup \{=_D\}$) and is obviously equivalent to N. Furthermore, from N' we can obtain a CSP instance N'' in CSP(Γ) by deleting constraints of the form $((v_i, v_j), =_D)$ and replacing any occurrence of v_j by v_i. Obviously, both transformation can be done in polynomial time. □

Reducibility IV

Corollary

A constraint language Γ is tractable if and only if its relational clone $\langle\Gamma\rangle$ is tractable. Γ is NP-complete if and only if $\langle\Gamma\rangle$ is NP-complete.

Remark: Γ is called NP-complete if CSP(Δ) is NP-complete for some finite subset $\Delta \subseteq \Gamma$.

Corollary

Let Γ be a constraint language and let R be a relation. R is expressible in Γ if and only if $R \in \langle\Gamma\rangle$.

Expressiveness

The indicator problem

Let $k \geq 1$ be a fixed natural number.
Let $s = (x_1, \ldots, x_m)$ be a list of k-tuples in D^k.
Let R be an n-ary relation on D.

We say, that s matches R if $n = m$ and if for each $1 \leq i \leq k$, the n-tuple $(x_1[i], \ldots, x_n[i])$ is in R.

Let now Γ be a fixed finite constraint language over a finite domain.
Set $I_k(\Gamma) = \langle V, D, C \rangle$, where

$$V := D^k$$
$$C := \{(s, R) : R \in \Gamma, s \text{ matches } R\}$$

Note: $I_k(\Gamma) \in \mathrm{CSP}(\Gamma)$ and contains constraints from Γ on every possible scope which matches some relation in Γ.

Definition

$I_k(\Gamma)$ is said to be the indicator problem of order k for Γ.

Example: $\neg, \oplus$

Consider the Boolean constraint language containing the unary relation $\neg$ and the exclusive-or relation $\oplus$, i.e.,

$$R_\oplus = \{(0,1),(1,0)\} \quad \text{and} \quad R_\neg = \{(0)\}.$$

The 3-rd order indicator problem of this language is:

1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

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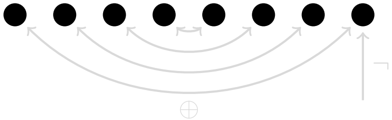
$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$



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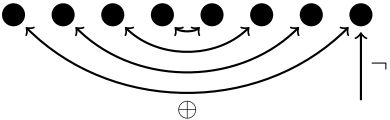
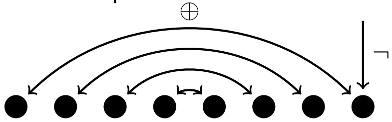


Example (cont'd): $\neg$, $\oplus$

Solutions of this indicator problem:



1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

Expressiveness and the indicator problem

Theorem (Jeavons (1998))

Let Γ be a finite constraint language over some finite domain D and let $R = \{t_1, \ldots, t_k\}$ be any n-ary relation on D.
Equivalent are:

(a) *R is expressible in Γ (i.e., $R \in \langle\Gamma\rangle$).*

(b) *$I_k(\Gamma)$ is a gadget for expressing R with construction site $(x_1, \ldots, x_n)$, where for each $1 \leq i \leq n$,*

$$x_i := (t_1[i], \ldots, t_k[i]).$$

Proof:

The direction from (b) to (a) is trivial, since $I_k(\Gamma)$ is contained in CSP(Γ). The other direction will be proved later. □

Example: $\neg, \oplus$

Problem: Is the implication expressible in the Boolean language $\{\neg, \oplus\}$?

Consider the 3rd indicator problem (since $R_\Rightarrow$ has three elements $(1,1),(0,0),(0,1)$). Consider the variables $v = (1,0,0)$ and $w = (1,0,1)$:

1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
Solutions							
1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
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1	1	0	1	0	1	0	0
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Example: $\neg, \oplus$

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Solutions							
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1	1	1	1	0	0	0	0
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1	1	**0**	**0**	1	1	0	0
1	0	**1**	**0**	1	0	1	0

Solutions

1	0	**1**	**1**	0	0	1	0
1	0	**1**	**0**	1	0	1	0
1	0	**0**	**1**	0	1	1	0
1	0	**0**	**0**	1	1	1	0
1	1	**1**	**1**	0	0	0	0
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Solutions							
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1	1	**1**	**0**	1	0	0	0
1	1	**0**	**1**	0	1	0	0
1	1	**0**	**0**	1	1	0	0

From this we obtain that $\pi_{(v,w)}(\mathrm{Sol}(I_3(\Gamma))) = D \times D \neq R_{\Rightarrow}$.

Thus, the implication is not expressible.

Polymorphisms

Let f be a k-ary operation, i.e., a function $f : D^k \to D$.
For any collection of n-tuples, $t_1, \dots, t_k \in D^n$, let $f(t_1, \dots, t_k)$ be defined as the n-tuple:

$$\big(f(t_1[1], \dots, t_k[1]), \dots, f(t_1[n], \dots, t_k[n])\big).$$

Definition

Let $f : D^k \to D$ be a k-ary operation, and R be an n-ary relation. f is a polymorphism of R (or: R is invariant under f) if for all $t_1, \dots, t_k \in R$, $f(t_1, \dots, t_k) \in R$.

Let Γ be a set of relations on a fixed domain D, and let F be a set of operations on D. Then define:

$\mathrm{Pol}(\Gamma)$:	the set of operations on D that preserve each relation in Γ
$\mathrm{Inv}(F)$:	the set of relations on D that are invariant under each operation of F

Lemma

Pol and Inv define anti-monotone functions, and are related by the following Galois correspondence:

$$\Gamma \subseteq \mathrm{Inv}(F) \iff F \subseteq \mathrm{Pol}(\Gamma).$$

In particular, it holds:

Indicator problem and polymorphisms

Lemma

Let Γ be a constraint language. The solutions of the k-th indicator problem $I_k(\Gamma)$ are precisely the k-ary polymorphisms of Γ.

Proof:

Apply the definitions . . . □

Expressiveness and polymorphisms

Lemma

Let Γ be a constraint language over some domain D. If $f : D^k \to D$ is a polymorphism of each $R \in \Gamma$, then f is a polymorphism of each $R \in \langle \Gamma \rangle$.

Proof:

Induction on primitive positive formula (exercise). $\square$

Expressiveness and the indicator problem (Part 2)

The following lemma completes the proof of Jeavons' theorem:

Lemma

Let $R = \{t_1, \dots, t_k\}$ be an n-ary relation (over some finite domain D). For $1 \leq i \leq n$, set $x_i := (t_1[i], \dots, t_k[i])$.
If R is expressible in Γ, then $R = \pi_{x_1,\dots,x_n}(\mathrm{Sol}(I_k(\Gamma)))$.

Proof:

Blackboard. □

Theorem

For any constraint language Γ over some finite domain D,

$$\langle\Gamma\rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$$

Proof:

$\subseteq$ is clear. For the converse let R be an n-ary relation that is invariant for each polymorphism of Γ. We have to show that $R \in \langle\Gamma\rangle$. Let $R = \{t_1, \ldots, t_k\}$ and consider the k-th indicator problem of Γ. First define $x_i := (t_1[i], \ldots, t_k[i])$ $(1 \leq i \leq n)$, then consider $R_t = \pi_{x_1,\ldots,x_n}(\mathrm{Sol}(I_k(\Gamma)))$. Obviously, R is expressible if $R = R_t$.
$R_t \subseteq R$ follows from the facts that every solution of $I_k(\Gamma)$ is a k-ary polymorphism and that each polymorphism of Γ preserves R.
For $R \subseteq R_t$, consider t_j in R. Now the j-th projection function $p_j : D^k \rightarrow D$ is a polymorphism, and hence a solution of $I_k(\Gamma)$. It follows $t_j = p_j(x_1, \ldots, x_n) \in R_t$. □

Expressiveness, Polymorphisms, and Complexity

Corollary

A relation R on a finite domain is expressible in a constraint language Γ if and only if $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\{R\})$.

Corollary

Let Γ and Δ be constraint languages on a finite domain. If Δ is finite and $\mathrm{Pol}(\Gamma) \subseteq \mathrm{Pol}(\Delta)$, then $CSP(\Delta)$ is polynomial-time reducible to $CSP(\Gamma)$.

Tractability over Finite Domains

Operations

Following, we study k-ary operations $f : D^k \to D$.

Definition

- f is idempotent if for each $x \in D$, $f(x, \ldots, x) = x$.
- Given $k = 3$, f is a majority operation if for all $x, y \in D$,

$$f(x,x,y) = f(x,y,x) = f(y,x,x) = x.$$

- Given $k = 3$, f is a Mal'tsev operation if for all $x, y \in D$,

$$f(y,y,x) = f(x,y,y) = x.$$

- f is conservative if for all $x_1, \ldots, x_k \in D$,

$$f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}.$$

Operations (cont'd)

Definition

- Given $k = 2$, f is a semi-lattice operation if it is
  - associative (i.e., $f(x, f(y,z)) = f(f(x,y),z)$),
  - commutative (i.e., $f(x,y) = f(y,x)$), and
  - idempotent.
- Given $k = 3$ and an Abelian group structure on D, f is affine if for all $x, y, z \in D$,

$$f(x,y,z) = x - y + z.$$

- Given $k \geq 3$, f is a near-unanimity operation if for all $x, y \in D$,

$$f(y,x,\ldots,x) = f(x,y,x\ldots,x) = \cdots = f(x,\ldots,x,y) = x.$$

Operations (cont'd)

Definition

- f is essentially unary if there exists an $1 \leq i \leq k$ and a unary non-constant operation g on D such that for all $x_1, \ldots, x_k \in D$,

$$f(x_1, \ldots, x_k) = g(x_i).$$

  If g is the identity operation, then f is called a projection.
- Given $k \geq 3$, f is a semi-projection if f is not a projection and there exists an $1 \leq i \leq k$, such that for all $x_1, \ldots, x_k \in D$ with $|\{x_1, \ldots, x_k\}| < k$,

$$f(x_1, \ldots, x_k) = x_i.$$

A necessary condition for tractability

Theorem

Given $P \neq NP$, any tractable constraint language Γ over a finite domain has a solution to an indicator problem $I_k(\Gamma)$ that defines

- *a constant operation,*
- *a majority operation,*
- *an idempotent binary operation,*
- *an affine operation, or*
- *a semi-projection.*

The complexity of any language over a domain of size 2 can be determined by considering the solutions of its 3rd order indicator problem. The problem is intractable unless this indicator problem has one of the following six solutions:

Variables									
1	1	1	1	0	0	0	0		
1	1	0	0	1	1	0	0		
1	0	1	0	1	0	1	0		
Solutions								**Schaefer class**	**Name**
0	0	0	0	0	0	0	0	1	Constant 0
1	1	1	1	1	1	1	1	2	Constant 1
1	1	1	1	1	1	1	0	3	Anti-Horn
1	0	0	0	0	0	0	0	4	Horn-SAT
1	1	1	0	1	0	0	0	5	2-SAT
1	0	0	1	0	1	1	0	6	Linear

Example: $\neg, \oplus$

●	●	●	●	●	●	●	●
1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
Solutions							
1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	**1**	**1**	**0**	**1**	**0**	**0**	**0**
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

Sufficient conditions: Semi-lattice operations

In what follows let Γ always be a constraint language over a finite domain D. We present some sufficient criteria for (in-) tractability.

Theorem

If $\mathrm{Pol}(\Gamma)$ *contains a semi-lattice operation, then*

- Γ *is tractable, and*
- *each instance of* $CSP(\Gamma)$ *can be solved by enforcing generalized arc consistency.*

Examples

Example 1:
If Γ is the Boolean constraint language containing relations expressible by conjunctions of Horn clauses, then

$$\wedge : \{0,1\}^2 \rightarrow \{0,1\}$$

is a semi-lattice operation that is a polymorphism of Γ.

Example 2:
If D is ordered, then max is a semi-lattice operation, which is a polymorphism of each set of max-closed relations.

Sufficient conditions: Conservative operations

Theorem

If Pol(Γ) *contains a conservative and commutative binary operation, then* Γ *is tractable.*

Note: If Γ contains all unary relations on D, then all operations in Pol(Γ) are conservative.

Sufficient conditions: Near-unanimity operations

Theorem

If Pol(Γ) *contains a k-ary near-unanimity operation, then*

- Γ *is tractable.*
- *Each instance of CSP*(Γ) *can be solved by enforcing strong k-consistency.*

Proof:

Blackboard. □

Examples

Example 3:

Let Γ be the Boolean constraint language that consists of relations definable by a PL-formula in CNF s. t. each conjunct has at most two literals.

Then

$$d(x,y,z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

is a near-unanimity operation on $\{0,1\}$ and a polym. of Γ.

Example 4:

The 0/1/all relations are invariant under the ternary operation

$$d(x,y,z) := \begin{cases} x & \text{if } y \neq z \\ y & \text{else} \end{cases}$$

which is a near-unanimity operation.

Sufficient conditions: Mal'tsev operations

Theorem

If Pol(Γ) *contains a k-ary Mal'tsev operation, then CSP*(Γ) *is tractable.*

Note: Affine relations are Mal'tsev operations.

Reduced constraint languages

Tractable Constraint Languages
Schaefer's Dichotomy Theorem
Relational Clones
Expressiveness
Polymorphisms
Tractability over Finite Domains
Literature

Lemma

Let Γ be a constraint language over D, and let f be a unary operation in $\mathrm{Pol}(\Gamma)$. Let $f(\Gamma)$ be the set of all $f(R) := \{f(t) : t \in R\}$ with $R \in \Gamma$. Then, CSP(Γ) is polynomial-time equivalent to CSP$(f(\Gamma))$.

Definition

A constraint language Γ is reduced if all its unary polymorphisms are surjective.

Note: Each constraint language can be transformed into a reduced language. For this find all unary polymorphisms by generating and solving the 1st order indicator problem. Choose one of these polymorphisms f with a minimal number of values in its range.

A sufficient condition for intractability

Theorem

Let Γ be a constraint language over a finite domain. If $\mathrm{Pol}(\Gamma)$ *contains only essentially unary operations, then CSP*(Γ) *is NP-complete.*

Proof idea:

We can assume that Γ is reduced. One can show that

- $\neq_D$ is in $\mathrm{Inv}(\mathrm{Pol}(\Gamma))$;
- if $|D| = 2$, $\mathrm{Inv}(\mathrm{Pol}(\Gamma))$ contains the not-all-equal relation:

$$D^3 \setminus \{(x,x,x) : x \in D\}$$

which ensures that CSP(Γ) intractable. □

It can be shown that for any reduced constraint language Γ on a finite domain D, one of the following conditions holds:

- $\mathrm{Pol}(\Gamma)$ contains a constant operation;
- $\mathrm{Pol}(\Gamma)$ contains a ternary near-unanimity operation;
- $\mathrm{Pol}(\Gamma)$ contains a Mal'tsev operation;
- $\mathrm{Pol}(\Gamma)$ contains an idempotent binary operation;
- $\mathrm{Pol}(\Gamma)$ contains a semi-projection;
- $\mathrm{Pol}(\Gamma)$ contains essentially unary operations only.

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