

Constraint Satisfaction Problems

Tractable Constraint Languages

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- For some restricted constraint languages we know some polynomial time algorithms that solve each instance of that language
 - Restricting constraint languages entails restricting expressiveness, i.e., the class of problems that can be expressed in the language
- ⇒ Expressiveness vs computational complexity?

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Definition

An **instance of a constraint satisfaction problem** (i.e., a **constraint network**) is a triple

$$N = \langle V, D, C \rangle,$$

where:

- V is a non-empty and finite set of **variables**,
- D is an arbitrary set (**domain**),
- C is a finite set of **constraints** C_1, \dots, C_q , i.e., each constraint C_i is a pair (s_i, R_i) , where s_i is a tuple of variables of length m_i and R_i is an m_i -ary relation on D (s_i : **constraint scope**; R_i : **constraint relation**).

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Restricting the general CSP



The **general CSP decision problem** is the following: Given an instance of a constraint satisfaction problem, N , determine if there exists solution to N , i.e., determine whether

$$\text{Sol}(N)$$

$$:= \{(d_1, \dots, d_n) \in D^n : a(v_i) = d_i \text{ for a solution } a \text{ of } N\}$$

(where n is the number of variables of V) is not empty.

Restricting the general CSP:

- **structural restriction**: consider just CSP instances with particular constraint scopes (e. g., where the network hypergraph has specific properties)
- **relational restriction**: consider just CSP instances, where the constraint relations have a specific form or specific properties

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Definition

A **constraint language** is an arbitrary set of relations, Γ , defined over some fixed domain (denoted by $D(\Gamma)$).

Definition

For a constraint language Γ , let $\text{CSP}(\Gamma)$ be the class of CSP instances $N = \langle V, D, C \rangle$ such that for each $(s, R) \in C$, $R \in \Gamma$. $\text{CSP}(\Gamma)$ is called the **relational subclass** associated with Γ .

Definition

A finite constraint language Γ is **tractable** if there exists a polynomial algorithm that solves all instances of $\text{CSP}(\Gamma)$.
An infinite constraint language Γ is **tractable** if each finite subset of the language is tractable.

Following we present some examples:

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Example: CHIP language



CHIP is a constraint language for arithmetic and other constraints.

Basic constraints in CHIP are so-called:

- **domain constraints:** unary constraints that restrict the domains of variables to a finite set of natural numbers
- **arithmetic constraints:** constraints of one of the forms

$$ax = by + c$$

$$ax \leq by + c$$

$$ax \geq by + c$$

$(a, b, c \in \mathbb{N}, a \neq 0)$. If these equations are conceived of as relations, the resulting constraint language is tractable.

The language is still tractable if we allow for relations expressed by

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \geq by + c$$

$$ax_1 \cdots x_n \geq by + c$$

$$(a_1x_1 \geq b_1) \vee \dots \vee (a_nx_n \geq b_n) \vee (ay \geq b)$$

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Example: Linear relations



Let D be any field (e.g., the field of real numbers).

A **linear relation** on D is any relation defined by some system of linear equations:

$$a_1x_1 + \dots + a_nx_n = r \quad (a_1, \dots, a_n, r \in D).$$

Then any instance of $\text{CSP}(\Gamma_{\text{lin}})$ can be represented by a system of linear equations over D , and hence can be solved in polynomial time (apply Gaussian elimination).

Hence, the language of all linear relations over D is tractable.

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Example: Relations on finite orderings

Let D be a finite ordered set.

Consider the binary **disequality relation**

$$\neq_D = \{(d_1, d_2) \in D^2 : d_1 \neq d_2\}$$

The class of CSP instances $\text{CSP}(\{\neq_D\})$ corresponds to the graph colorability problem with $|D|$ colors.

$\text{CSP}(\{\neq_D\})$ is tractable if $|D| \leq 2$ or $|D| = \infty$, and intractable, otherwise.

The ternary **betweenness relation** over D is defined by:

$$B_D = \{(a, b, c) \in D^3 : a < b < c \vee c < b < a\}$$

$\text{CSP}(\{B_D\})$ is tractable if $|D| \leq 4$, and intractable if $|D| \geq 5$.

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Example: Connected row-convex relations



Let $D = \{d_1, \dots, d_n\}$ be a finite (totally) ordered set.

For a binary relation R over D , the matrix representation of R is an $n \times n$ 0,1-matrix M_R , where $M_R[d, d'] = 1$ iff $(d, d') \in R$.

The **pruned matrix representation** of R results from the matrix representation of R , when we remove all rows and columns in which only 0's occur.

R is **connected row-convex**, if in the pruned matrix representation of R , the pattern of 1's is connected along each column, along each row, and forms a connected 2-dimensional region.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The constraint language on any class of connected row-convex

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Example: Boolean constraints

Let $D = \{0, 1\}$.

The class of CSP instances $\text{CSP}(\{N_D\})$, where

$$N_D = D^3 \setminus \{(0, 0, 0), (1, 1, 1)\}$$

is the **not-all-equal relation** over D , is intractable.

$\text{CSP}(\{N_D\})$ corresponds to the not-all-equal satisfiability problem (NAE-3SAT), which is known to be NP-hard.

The class of CSP instances $\text{CSP}(\{T_D\})$, where

$$T_D = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\},$$

is intractable.

$\text{CSP}(\{T_D\})$ corresponds to the one-in-three satisfiability problem (1-in-3 SAT).

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Example: 0/1/all-relations



Let D be an arbitrary finite set. A relation R over D is called **0/1/all-relation** if one of the following conditions holds:

- R is unary;
- $R = D_1 \times D_2$ for subsets D_1, D_2 of D ;
- $R = \{(d, \pi(d)) : d \in D_1\}$, for some subset $D_1 \subseteq D$ and some permutation π of D ;
- $R = \{(a, b) \in D_1 \times D_2 : a = d_1 \vee b = d_2\}$, for some subsets D_1, D_2 of D and some elements $d_1 \in D_1, d_2 \in D_2$.

The language defined by all 0/1/all-relations is tractable.

It is even **maximal tractable**: if we add any binary relation over D that is not a 0/1/all-relation, then the resulting constraint language becomes intractable.

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Let $(D, <)$ be a linear order. Define $\max : D \times D \rightarrow D$ in the usual way, i.e., $\max(a, b) = a$ if $a > b$, and $\max(a, b) = b$, otherwise.

We extend \max to a function that can be applied to tuples, i.e., we define $\max : D^k \times D^k \rightarrow D^k$ by

$$\begin{aligned} \max((a_1, \dots, a_k), (b_1, \dots, b_k)) \\ := (\max(a_1, b_1), \dots, \max(a_k, b_k)). \end{aligned}$$

Definition

An n -ary relation R over D is **max-closed** if for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in R$,

$$\max((a_1, \dots, a_n), (b_1, \dots, b_n)) \in R.$$

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Lemma

*Let Γ be a constraint language with max-closed relations only.
Then $CSP(\Gamma)$ is tractable.*

Proof:

Enforce generalized arc consistency. If any domain of the resulting network is empty, the network is inconsistent. Otherwise, set each variable to its maximal value, □

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Example: max-closed relations

Consider the CHIP language. All relations of CHIP are max-closed. Hence any set of equations can be solved by establishing gen. arc consistency.

For example, consider a CSP instance with domain $\{1, \dots, 5\}$, variables $\{v, w, x, y, z\}$, and equations

$$w \neq 3, z \neq 5, 3v \leq z, y \geq z + 2,$$

$$3x + y + z \geq 5w + 1, wz \geq 2y.$$

Enforcing gen. arc consistency results in:

$$D(v) = \{1\}, D(w) = \{4\}, D(x) = \{4, 5\},$$

$$D(y) = \{5\}, D(z) = \{3\}.$$

Hence

$$v \mapsto 1, w \mapsto 4, x \mapsto 5, y \mapsto 5, z \mapsto 3$$

is a solution of the constraint network.

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2 Schaefer's Dichotomy Theorem



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The key result in the literature on tractable constraint languages is Schaefer's Dichotomy Theorem (1978).

Definition

A **Boolean constraint language** is a constraint language over the two-element domain $D = \{0, 1\}$.

Schaefer's theorem states that any Boolean constraint language is either tractable or NP-complete. Moreover, it provides a classification of all tractable constraint languages.

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Theorem (Schaefer 1978)

Let Γ be a Boolean constraint language. Then Γ is tractable if at least one of the following conditions is satisfied:

- 1 Each relation in Γ contains the tuple $(0, \dots, 0)$.*
- 2 Each relation in Γ contains the tuple $(1, \dots, 1)$.*
- 3 Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one negative literal.*
- 4 Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one positive literal.*
- 5 Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most two literals.*
- 6 Each relation in Γ is the set of solutions of a system of linear equations over the finite field with 2 elements.*

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Let Γ be a Boolean constraint language.

Class 1: any CSP instance N can be solved by simply assigning 0 to each variable of N .

Class 2: cf. Class 1 ($v \mapsto 1$).

Class 6: any CSP instance N can be solved by applying the Gaussian elimination procedure.

Class 5: any CSP instance N can be solved by resolution: in this case $\text{CSP}(\Gamma)$ corresponds to the 2-SAT satisfiability problem and this can be solved efficiently by resolution.

Class 4: any CSP instance N can be solved by unit resolution: here $\text{CSP}(\Gamma)$ corresponds to the Horn-SAT satisfiability problem, which can be solved efficiently by unit resolution.

Class 3: cf. Class 4 (“anti-Horn”).

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Definition

Let Γ be constraint language and R be a relation on $D(\Gamma)$.

R is **expressible** in Γ if there exists a CSP instance $N \in \text{CSP}(\Gamma)$ and a sequence of variables x_1, \dots, x_r in N such that

$$R = \pi_{x_1, \dots, x_r}(\text{Sol}(N)).$$

N is referred to as a **gadget** for expressing R in $\text{CSP}(\Gamma)$, the sequence x_1, \dots, x_r as **construction site** for R .

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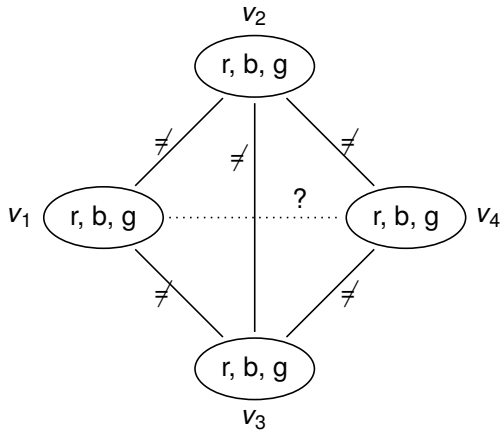
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Which relation is expressed by the edge (v_1, v_4) ?

Expressiveness can also be reformulated in the following way:
Let Γ, Γ' be constraint languages (def. on the same domain D).

Definition

Γ' is a **relational clone** of Γ if Γ' contains each relation definable by a FO-formula with

- relations from $\Gamma \cup \{=_D\}$,
- conjunctions, and
- existential quantification.

(Formulae of this form are called **primitive positive formulae**.)

Definition

Let Γ be a constraint language. $\langle \Gamma \rangle$ denotes the smallest relational clone containing Γ , **the clone generated by Γ** .

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Example

Consider a Boolean constraint language with the following relations:

$$R_1 = \{(0, 1), (1, 0), (1, 1)\} \quad R_2 = \{(0, 0), (0, 1), (1, 0)\}.$$

The relational clone generated by the set of these two relations contains all 16 binary Boolean relations. For example:

$$\begin{array}{ll} R_3 := \{(0, 1), (1, 0)\} & R_1(v_1, v_2) \wedge R_2(v_1, v_2) \\ R_4 := \{(0, 0), (1, 0), (1, 1)\} & \exists y(R_1(v_1, y) \wedge R_2(y, v_2)) \\ R_5 := \{(0, 0), (1, 1)\} & v_1 = v_2 \\ R_6 := \{(0, 0)\} & R_2(v_1, v_2) \wedge R_5(v_1, v_2) \\ R_7 := \{(1, 1)\} & R_1(v_1, v_2) \wedge R_5(v_1, v_2) \\ R_8 := \{(0, 1)\} & \exists y(R_6(v_1, y) \wedge R_1(y, v_2)) \end{array}$$

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Theorem

Let Γ be a set of relations on a fixed domain D , and let Δ be a finite subset of $\langle \Gamma \rangle$. Then there exists a polynomial time reduction from $CSP(\Delta)$ to $CSP(\Gamma)$.

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Proof:

Let $\Delta = \{S_1, \dots, S_k\}$ be a finite set of relations, where each S_j is definable by a pp-formula with relations from Γ and the relation $=_D$. For each S_j fix such a formula $\varphi_j(x_1, \dots, x_{r_j})$, where r_j is the arity of S_j . Without loss of generality, we may assume that each $\varphi_j(x_1, \dots, x_{r_j})$ has the form

$$\exists u_1 \dots u_m (R_1(w_1^1, \dots, w_{k_1}^1) \wedge \dots \wedge R_n(w_1^n, \dots, w_{k_n}^n)) \quad (1)$$

where $w_1^1, \dots, w_{k_1}^1, \dots, w_1^n, \dots, w_{k_n}^n \in \{x_1, \dots, x_{r_j}, u_1, \dots, u_m\}$ for some auxiliary variables u_1, \dots, u_m , and $R_1, \dots, R_n \in \Gamma \cup \{=_D\}$

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Let $N = \langle V, D, C \rangle$ be an arbitrary instance in $\text{CSP}(\Delta)$. Initially, set $V' := V, D' := D, C' := C$. For each constraint (s, R) (where $s = (v_1, \dots, v_r)$) of N , proceed as follows:

- 1 add the auxiliary variables u_1, \dots, u_m to V' (always add new variables, rename variables if necessary (also in (1)))
- 2 remove (r, R) from C' and instead add to C' the constraints (cf. (1)):

$$((w_1^1, \dots, w_{k_1}^1), R_1), \dots, (w_1^n, \dots, w_{k_n}^n), R_n)$$

The CSP instance N' obtained by this procedure is contained in $\text{CSP}(\Gamma \cup \{=_D\})$ and is obviously equivalent to N . Furthermore, from N' we can obtain a CSP instance N'' in $\text{CSP}(\Gamma)$ by deleting constraints of the form $((v_i, v_j), =_D)$ and replacing any occurrence of v_j by v_i . Obviously, both transformation can be done in polynomial time. \square

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Corollary

A constraint language Γ is tractable if and only if its relational clone $\langle \Gamma \rangle$ is tractable. Γ is NP-complete if and only if $\langle \Gamma \rangle$ is NP-complete.

Remark: Γ is called **NP-complete** if $\text{CSP}(\Delta)$ is NP-complete for some finite subset $\Delta \subseteq \Gamma$.

Corollary

Let Γ be a constraint language and let R be a relation. R is expressible in Γ if and only if $R \in \langle \Gamma \rangle$.

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The indicator problem



Let $k \geq 1$ be a fixed natural number.

Let $s = (x_1, \dots, x_m)$ be a list of k -tuples in D^k .

Let R be an n -ary relation on D .

We say, that s **matches** R if $n = m$ and if for each $1 \leq i \leq k$, the n -tuple $(x_1[i], \dots, x_n[i])$ is in R .

Let now Γ be a fixed finite constraint language over a finite domain.

Set $I_k(\Gamma) = \langle V, D, C \rangle$, where

$$V := D^k$$

$$C := \{(s, R) : R \in \Gamma, s \text{ matches } R\}$$

Note: $I_k(\Gamma) \in \text{CSP}(\Gamma)$ and contains constraints from Γ on every possible scope which matches some relation in Γ .

Definition

$I_k(\Gamma)$ is said to be the **indicator problem of order k** for Γ .

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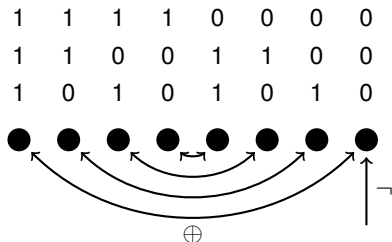
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Example: \neg, \oplus

Consider the Boolean constraint language containing the unary relation \neg and the exclusive-or relation \oplus , i.e.,

$$R_{\oplus} = \{(0, 1), (1, 0)\} \quad \text{and} \quad R_{\neg} = \{(0)\}.$$

The 3-rd order indicator problem of this language is:



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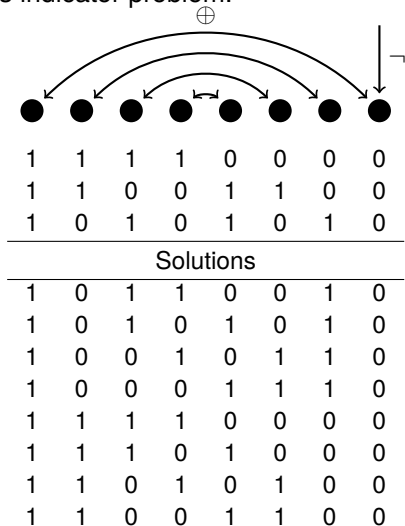
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Example (cont'd): \neg, \oplus



Solutions of this indicator problem:



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Theorem (Jeavons (1998))

Let Γ be a finite constraint language over some finite domain D and let $R = \{t_1, \dots, t_k\}$ be any n -ary relation on D .

Equivalent are:

- (a) R is expressible in Γ (i.e., $R \in \langle \Gamma \rangle$).
- (b) $I_k(\Gamma)$ is a gadget for expressing R with construction site (x_1, \dots, x_n) , where for each $1 \leq i \leq n$,

$$x_i := (t_1[i], \dots, t_k[i]).$$

Proof:

The direction from (b) to (a) is trivial, since $I_k(\Gamma)$ is contained in $\text{CSP}(\Gamma)$. The other direction will be proved later. □

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Example: \neg, \oplus

Problem: Is the implication expressible in the Boolean language $\{\neg, \oplus\}$?

Consider the 3rd indicator problem (since R_{\Rightarrow} has three elements $(1, 1), (0, 0), (0, 1)$). Consider the variables $v = (1, 0, 0)$ and $w = (1, 0, 1)$:

1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

From this we obtain that $\pi_{(v,w)}(\text{Sol}(I_3(\Gamma))) = D \times D \neq R_{\Rightarrow}$.

Thus, the implication is not expressible.

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Let f be a k -ary operation, i.e., a function $f : D^k \rightarrow D$.

For any collection of n -tuples, $t_1, \dots, t_k \in D^n$, let $f(t_1, \dots, t_k)$ be defined as the n -tuple:

$$(f(t_1[1], \dots, t_k[1]), \dots, f(t_1[n], \dots, t_k[n])).$$

Definition

Let $f : D^k \rightarrow D$ be a k -ary operation, and R be an n -ary relation. f is a **polymorphism** of R (or: R is **invariant** under f) if for all $t_1, \dots, t_k \in R$, $f(t_1, \dots, t_k) \in R$.

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Let Γ be a set of relations on a fixed domain D , and let F be a set of operations on D . Then define:

$\text{Pol}(\Gamma)$: the set of operations on D that preserve each relation in Γ

$\text{Inv}(F)$: the set of relations on D that are invariant under each operation of F

Lemma

Pol and Inv define anti-monotone functions, and are related by the following Galois correspondence:

$$\Gamma \subseteq \text{Inv}(F) \iff F \subseteq \text{Pol}(\Gamma).$$

In particular, it holds:

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Lemma

Let Γ be a constraint language. The solutions of the k -th indicator problem $I_k(\Gamma)$ are precisely the k -ary polymorphisms of Γ .

Proof:

Apply the definitions . . .



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Lemma

Let Γ be a constraint language over some domain D . If $f : D^k \rightarrow D$ is a polymorphism of each $R \in \Gamma$, then f is a polymorphism of each $R \in \langle \Gamma \rangle$.

Proof:

Induction on primitive positive formula (exercise). □

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Expressiveness and the indicator problem

(Part 2)



The following lemma completes the proof of Jeavons' theorem:

Lemma

Let $R = \{t_1, \dots, t_k\}$ be an n -ary relation (over some finite domain D). For $1 \leq i \leq n$, set $x_i := (t_1[i], \dots, t_k[i])$.

If R is expressible in Γ , then $R = \pi_{x_1, \dots, x_n}(\text{Sol}(I_k(\Gamma)))$.

Proof:

Blackboard.



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Theorem

For any constraint language Γ over some finite domain D ,

$$\langle \Gamma \rangle = \text{Inv}(\text{Pol}(\Gamma))$$

Proof:

\subseteq is clear. For the converse let R be an n -ary relation that is invariant for each polymorphism of Γ . We have to show that $R \in \langle \Gamma \rangle$. Let

$R = \{t_1, \dots, t_k\}$ and consider the k -th indicator problem of Γ . First define $x_i := (t_1[i], \dots, t_k[i])$ ($1 \leq i \leq n$), then consider

$R_t = \pi_{x_1, \dots, x_n}(\text{Sol}(I_k(\Gamma)))$. Obviously, R is expressible if $R = R_t$.

$R_t \subseteq R$ follows from the facts that every solution of $I_k(\Gamma)$ is a k -ary polymorphism and that each polymorphism of Γ preserves R .

For $R \subseteq R_t$, consider t_j in R . Now the j -th projection function $p_j : D^k \rightarrow D$ is a polymorphism, and hence a solution of $I_k(\Gamma)$. It follows

$t_j = p_j(x_1, \dots, x_n) \in R_t$. □

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Corollary

A relation R on a finite domain is expressible in a constraint language Γ if and only if $\text{Pol}(\Gamma) \subseteq \text{Pol}(\{R\})$.

Corollary

Let Γ and Δ be constraint languages on a finite domain. If Δ is finite and $\text{Pol}(\Gamma) \subseteq \text{Pol}(\Delta)$, then $\text{CSP}(\Delta)$ is polynomial-time reducible to $\text{CSP}(\Gamma)$.

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Following, we study k -ary operations $f : D^k \rightarrow D$.

Definition

- f is **idempotent** if for each $x \in D$, $f(x, \dots, x) = x$.
- Given $k = 3$, f is a **majority operation** if for all $x, y \in D$,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x.$$

- Given $k = 3$, f is a **Mal'tsev operation** if for all $x, y \in D$,

$$f(y, y, x) = f(x, y, y) = x.$$

- f is **conservative** if for all $x_1, \dots, x_k \in D$,

$$f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}.$$

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Definition

- Given $k = 2$, f is a **semi-lattice operation** if it is
 - associative (i.e., $f(x, f(y, z)) = f(f(x, y), z)$),
 - commutative (i.e., $f(x, y) = f(y, x)$), and
 - idempotent.
- Given $k = 3$ and an Abelian group structure on D , f is **affine** if for all $x, y, z \in D$,

$$f(x, y, z) = x - y + z.$$

- Given $k \geq 3$, f is a **near-unanimity operation** if for all $x, y \in D$,

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y) = x.$$

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Definition

- f is **essentially unary** if there exists an $1 \leq i \leq k$ and a unary non-constant operation g on D such that for all $x_1, \dots, x_k \in D$,

$$f(x_1, \dots, x_k) = g(x_i).$$

If g is the identity operation, then f is called a **projection**.

- Given $k \geq 3$, f is a **semi-projection** if f is not a projection and there exists an $1 \leq i \leq k$, such that for all $x_1, \dots, x_k \in D$ with $|\{x_1, \dots, x_k\}| < k$,

$$f(x_1, \dots, x_k) = x_i.$$

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Theorem

Given $P \neq NP$, any tractable constraint language Γ over a finite domain has a solution to an indicator problem $I_k(\Gamma)$ that defines

- *a constant operation,*
- *a majority operation,*
- *an idempotent binary operation,*
- *an affine operation, or*
- *a semi-projection.*

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The complexity of any language over a domain of size 2 can be determined by considering the solutions of its 3rd order indicator problem. The problem is intractable unless this indicator problem has one of the following six solutions:

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Variables								Schaefer class	Name
Solutions									
1	1	1	1	0	0	0	0	1	Constant 0
1	1	0	0	1	1	0	0	2	Constant 1
1	1	1	1	1	1	1	0	3	Anti-Horn
1	0	0	0	0	0	0	0	4	Horn-SAT
1	1	1	0	1	0	0	0	5	2-SAT
1	0	0	1	0	1	1	0	6	Linear

Example: \neg, \oplus



●	●	●	●	●	●	●	●
1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
<hr/>							
Solutions							
<hr/>							
1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

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In what follows let Γ always be a constraint language over a finite domain D . We present some sufficient criteria for (in-) tractability.

Theorem

If $\text{Pol}(\Gamma)$ contains a semi-lattice operation, then

- *Γ is tractable, and*
- *each instance of $\text{CSP}(\Gamma)$ can be solved by enforcing generalized arc consistency.*

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Example 1:

If Γ is the Boolean constraint language containing relations expressible by conjunctions of **Horn clauses**, then

$$\wedge : \{0, 1\}^2 \rightarrow \{0, 1\}$$

is a semi-lattice operation that is a polymorphism of Γ .

Example 2:

If D is ordered, then \max is a semi-lattice operation, which is a polymorphism of each set of max-closed relations.

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Theorem

If $\text{Pol}(\Gamma)$ contains a conservative and commutative binary operation, then Γ is tractable.

Note: If Γ contains all unary relations on D , then all operations in $\text{Pol}(\Gamma)$ are conservative.

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Sufficient conditions: Near-unanimity operations



Theorem

If $\text{Pol}(\Gamma)$ contains a k -ary near-unanimity operation, then

- Γ is tractable.
- Each instance of $\text{CSP}(\Gamma)$ can be solved by enforcing strong k -consistency.

Proof:

Blackboard. □

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Example 3:

Let Γ be the Boolean constraint language that consists of relations definable by a PL-formula in CNF s. t. each conjunct has at most two literals.

Then

$$d(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

is a near-unanimity operation on $\{0, 1\}$ and a polym. of Γ .

Example 4:

The 0/1/all relations are invariant under the ternary operation

$$d(x, y, z) := \begin{cases} x & \text{if } y \neq z \\ y & \text{else} \end{cases}$$

which is a near-unanimity operation.

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Theorem

If $\text{Pol}(\Gamma)$ contains a k -ary Mal'tsev operation, then $\text{CSP}(\Gamma)$ is tractable.

Note: Affine relations are Mal'tsev operations.

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Lemma

Let Γ be a constraint language over D , and let f be a unary operation in $\text{Pol}(\Gamma)$. Let $f(\Gamma)$ be the set of all $f(R) := \{f(t) : t \in R\}$ with $R \in \Gamma$. Then, $\text{CSP}(\Gamma)$ is polynomial-time equivalent to $\text{CSP}(f(\Gamma))$.

Definition

A constraint language Γ is **reduced** if all its unary polymorphisms are surjective.

Note: Each constraint language can be transformed into a reduced language. For this find all unary polymorphisms by generating and solving the 1st order indicator problem. Choose one of these polymorphisms f with a minimal number of values in its range.

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Theorem

Let Γ be a constraint language over a finite domain. If $\text{Pol}(\Gamma)$ contains only essentially unary operations, then $\text{CSP}(\Gamma)$ is NP-complete.

Proof idea:

We can assume that Γ is reduced. One can show that

- \neq_D is in $\text{Inv}(\text{Pol}(\Gamma))$;
- if $|D| = 2$, $\text{Inv}(\text{Pol}(\Gamma))$ contains the not-all-equal relation:

$$D^3 \setminus \{(x, x, x) : x \in D\}$$

which ensures that $\text{CSP}(\Gamma)$ intractable. □

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It can be shown that for any reduced constraint language Γ on a finite domain D , one of the following conditions holds:

- $\text{Pol}(\Gamma)$ contains a constant operation;
- $\text{Pol}(\Gamma)$ contains a ternary near-unanimity operation;
- $\text{Pol}(\Gamma)$ contains a Mal'tsev operation;
- $\text{Pol}(\Gamma)$ contains an idempotent binary operation;
- $\text{Pol}(\Gamma)$ contains a semi-projection;
- $\text{Pol}(\Gamma)$ contains essentially unary operations only.

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