# Constraint Satisfaction Problems

## Constraint Optimization

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

**Stefan Wölfl**, **Christian Becker-Asano**, and **Bernhard Nebel**

January 7, 2015

# Motivation

# Hard and soft constraints



Real-life problems often contain hard and soft constraints:

Hard constraints: must be satisfied;

Soft constraints: should be satisfied, but may be violated.

Example: In time-tabling problems,

- resource constraints such as “a teacher can teach only one class at a time” must be satisfied;
- a request such as “the schedule of teacher should be concentrated in two days” is simply a preference, but not essential for the solution.

What to do with soft constraints?

# Hard and soft constraints



Real-life problems often contain hard and soft constraints:

Hard constraints: must be satisfied;

Soft constraints: should be satisfied, but may be violated.

Example: In time-tabling problems,

- resource constraints such as “a teacher can teach only one class at a time” must be satisfied;
- a request such as “the schedule of teacher should be concentrated in two days” is simply a preference, but not essential for the solution.

What to do with soft constraints?

# Hard and soft constraints





Real-life problems often contain hard and soft constraints:

Hard constraints: must be satisfied;

Soft constraints: should be satisfied, but may be violated.

Example: In time-tabling problems,

- resource constraints such as “a teacher can teach only one class at a time” must be satisfied;
- a request such as “the schedule of teacher should be concentrated in two days” is simply a preference, but not essential for the solution.

What to do with soft constraints?

# Constraint optimization

Formalizing problems with soft and hard constraints leads to constraint networks augmented with a global cost function (also called criterion function or objective function), based on the satisfaction of soft constraints.

A constraint optimization problem (COP) is the problem of finding a variable assignment to all variables that satisfies all hard constraints and at the same time optimizes the global cost function.

Note: Every constraint satisfaction problem can be viewed as a constraint optimization problem – when not all constraints are satisfiable. Try to find an assignment that maximizes the number of satisfied constraints: MAX-CSP problem.

# Constraint optimization



Formalizing problems with soft and hard constraints leads to constraint networks augmented with a global cost function (also called criterion function or objective function), based on the satisfaction of soft constraints.

A constraint optimization problem (COP) is the problem of finding a variable assignment to all variables that satisfies all hard constraints and at the same time optimizes the global cost function.

Note: Every constraint satisfaction problem can be viewed as a constraint optimization problem – when not all constraints are satisfiable. Try to find an assignment that maximizes the number of satisfied constraints: MAX-CSP problem.

# Constraint optimization



Formalizing problems with soft and hard constraints leads to constraint networks augmented with a global cost function (also called criterion function or objective function), based on the satisfaction of soft constraints.

A constraint optimization problem (COP) is the problem of finding a variable assignment to all variables that satisfies all hard constraints and at the same time optimizes the global cost function.

Note: Every constraint satisfaction problem can be viewed as a constraint optimization problem – when not all constraints are satisfiable. Try to find an assignment that maximizes the number of satisfied constraints: MAX-CSP problem.

# Example 1: Power plant maintenance



Given

1. a number of power generators,
2. preventive maintenance intervals,
3. time for maintenance,
4. accurate estimates for plant's power demands,

determine a maintenance schedule respecting (2) that minimizes operating and maintenance costs.

# Example 1: Power plant maintenance



Given

1. a number of power generators,
2. preventive maintenance intervals,
3. time for maintenance,
4. accurate estimates for plant's power demands,

determine a maintenance schedule respecting (2) that minimizes operating and maintenance costs.

# Example 2: Combinatorial auctions



In combinatorial auctions, bidders can give bids for sets of items. The auctioneer then has to generate an optimal selection, e.g., one that maximizes revenue.

**Definition**

The combinatorial auction problem is specified as follows:

Given: A set of items $Q = \{q_1, \ldots, q_n\}$ and a set of bids $B = \{b_1, \ldots, b_m\}$ such that each bid is $b_i = (Q_i, r_i)$, where $Q_i \subseteq Q$ and $r_i$ is a strictly positive real number.

Task: Find a subset of bids $B' \subseteq B$ such that any two bids in $B'$ do not share an item maximizing $\sum_{(Q_i, r_i) \in B'} r_i$.

# Example 2: Combinatorial auctions



In combinatorial auctions, bidders can give bids for sets of items. The auctioneer then has to generate an optimal selection, e.g., one that maximizes revenue.

## Definition

The combinatorial auction problem is specified as follows:

**Given:** A set of items $Q = \{q_1, \ldots, q_n\}$ and a set of bids $B = \{b_1, \ldots, b_m\}$ such that each bid is $b_i = (Q_i, r_i)$, where $Q_i \subseteq Q$ and $r_i$ is a strictly positive real number.

**Task:** Find a subset of bids $B' \subseteq B$ such that any two bids in $B'$ do not share an item maximizing $\sum_{(Q_i, r_i) \in B'} r_i$.

# Cost Networks

# From constraint to cost networks



- We will extend constraint networks to cost networks.
- Hard constraints are modelled as ordinary constraints, we know already.
- Soft constraints are modelled by cost functions, which assign particular costs to variable assignments.
- The costs are aggregated by a global cost function

# Global cost functions



A constraint optimization problem (COP) is a constraint network extended by a global cost function.

**Definition**

Given a set of variables $V = \{v_1, \ldots, v_n\}$, a set of real-valued functions $F_1, \ldots, F_l$ over scopes $s_1, \ldots, s_l$ ($s_j \subseteq V$), and assignments $a$ over $V$. The global cost function $F$ is defined by

$$F(a) = \sum_{j=1}^{l} F_j(a),$$

where $F_j(a)$ means $F_j$ applied to assignment $a$ restricted to the scope of $F_j$, i.e., $F_j(a) = F_j(a[s_j])$.

# Cost network



Constraint optimization problems can be viewed as defined over an extended constraint network called cost network.

## Definition

A cost network is a 4-tuple $\mathcal{C} = \langle V, D, C_h, C_s \rangle$, where

(a) $\langle V, D, C_h \rangle$ is a constraint network (elements of $C_h$ are called hard constraints), and

(b) $C_s = \{F_1, \ldots, F_l\}$ is a set of real-valued functions defined over scopes $s_1, \ldots, s_l$ (elements of $C_s$ are called soft constraints).

## Definition

A solution to a constraint optimization problem given by a cost network $\mathcal{C} = \langle V, D, C_h, C_s \rangle$, is an assignment $a^*$ that maximizes (minimizes) $F(a)$ among all assignments $a$ that satisfy $\langle V, D, C_h \rangle$.

# Cost network



Constraint optimization problems can be viewed as defined over an extended constraint network called cost network.

## Definition

A cost network is a 4-tuple $\mathscr{O} = \langle V, D, C_h, C_s \rangle$, where

(a) $\langle V, D, C_h \rangle$ is a constraint network (elements of $C_h$ are called hard constraints), and

(b) $C_s = \{F_1, \ldots, F_l\}$ is a set of real-valued functions defined over scopes $s_1, \ldots, s_l$ (elements of $C_s$ are called soft constraints).

## Definition

A solution to a constraint optimization problem given by a cost network $\mathscr{O} = \langle V, D, C_h, C_s \rangle$, is an assignment $a^*$ that maximizes (minimizes) $F(a)$ among all assignments $a$ that satisfy $\langle V, D, C_h \rangle$.

# Example: Cost network for combinatorial auction



For a combinatorial auction given by item set $Q = \{q_1, \ldots, q_n\}$ and bids $B = \{b_1, \ldots, b_m\}$ with $b_i = (Q_i, r_i)$ define a cost network as follows:

- Variables $b_i$ with domain $\{0,1\}$; 1 for selecting the bid, 0 otherwise;
- For each pair $b_i, b_j$ such that $Q_i \cap Q_j \neq \emptyset$ a constraint $R_{ij}$ prohibiting that $b_i$ and $b_j$ are assigned 1 simultaneously;
- Cost functions $F_i$ with $F_i(a) = r_i$ if $a(b_i) = 1$, $F_i(a) = 0$ otherwise, for an assignment $a$.

Find a consistent assignment $a$ to the $b_i$ s that maximizes $F(a) = \sum_i F_i(a)$.

Note: cost network = constraint network, because all cost components are unary.

# Example: Auction



Consider the following auction:

$$
\begin{aligned}
b_1: \quad Q_1 &= \{1,2,3,4\}, & r_1 &= 8, \\
b_2: \quad Q_2 &= \{2,3,6\}, & r_2 &= 6, \\
b_3: \quad Q_3 &= \{1,4,5\}, & r_3 &= 5, \\
b_4: \quad Q_4 &= \{2,8\}, & r_4 &= 2, \\
b_5: \quad Q_5 &= \{5,6\}, & r_5 &= 2.
\end{aligned}
$$

What is the optimal assignment?

# Example: Auction



Consider the following auction:

$$
\begin{array}{llllll}
b_1: & Q_1 & = & \{1,2,3,4\}, & r_1 & = & 8, \\
b_2: & Q_2 & = & \{2,3,6\}, & r_2 & = & 6, \\
b_3: & Q_3 & = & \{1,4,5\}, & r_3 & = & 5, \\
b_4: & Q_4 & = & \{2,8\}, & r_4 & = & 2, \\
b_5: & Q_5 & = & \{5,6\}, & r_5 & = & 2.
\end{array}
$$

What is the optimal assignment?

# Reduction of COP-solving to CSP-solving

We can always reduce COP-solving to solving a sequence of CSPs.

Given a COP $\mathscr{O}$ which we want to maximize. Consider a sequence of CSPs $\mathscr{C}_i$, s. t. each contains the constraint part of $\mathscr{O}$ and an additional constraint $\sum_j F_j(a) \geq c_i$, where $c_1 \leq \ldots \leq c_i \leq \ldots$.

Solve the CSPs with increasing cost bounds $c_i$ until no solution can be found. Then the previous step is the optimal solution – provided the difference between the steps is not larger than the smallest difference between different values of the global cost function.

# Reduction of COP-solving to CSP-solving



We can always reduce COP-solving to solving a sequence of CSPs.

Given a COP $\mathscr{O}$ which we want to maximize. Consider a sequence of CSPs $\mathscr{C}_i$, s. t. each contains the constraint part of $\mathscr{O}$ and an additional constraint $\sum_j F_j(a) \geq c_i$, where $c_1 \leq \ldots \leq c_i \leq \ldots$.

Solve the CSPs with increasing cost bounds $c_i$ until no solution can be found. Then the previous step is the optimal solution – provided the difference between the steps is not larger than the smallest difference between different values of the global cost function.

# Example: Solving the auction problem



Assumption: Step size 1 and static variable ordering $b_1, b_2, b_3, b_4, b_5$.

For cost bounds from $c_1 = 0$ to $c_9 = 8$, $a(b_1) = 1$ and all others 0 is satisfying.

For cost bound $c_{10} = 9$ and $c_{11} = 10$, $a(b_1) = 1$ and $a(b_5) = 1$ (and all others 0) is satisfying.

For cost bound $c_{12} = 11$, $a(b_2) = 1$ and $a(b_3) = 1$ (and all others 0) is satisfying.

For cost bound $c_{13} = 12$, there is no satisfying assignment.

# Example: Solving the auction problem



Assumption: Step size 1 and static variable ordering $b_1, b_2, b_3, b_4, b_5$.

For cost bounds from $c_1 = 0$ to $c_9 = 8$, $a(b_1) = 1$ and all others 0 is satisfying.

For cost bound $c_{10} = 9$ and $c_{11} = 10$, $a(b_1) = 1$ and $a(b_5) = 1$ (and all others 0) is satisfying.

For cost bound $c_{12} = 11$, $a(b_2) = 1$ and $a(b_3) = 1$ (and all others 0) is satisfying.

For cost bound $c_{13} = 12$, there is no satisfying assignment.

# Example: Solving the auction problem



Assumption: Step size 1 and static variable ordering $b_1, b_2, b_3, b_4, b_5$.

For cost bounds from $c_1 = 0$ to $c_9 = 8$, $a(b_1) = 1$ and all others 0 is satisfying.

For cost bound $c_{10} = 9$ and $c_{11} = 10$, $a(b_1) = 1$ and $a(b_5) = 1$ (and all others 0) is satisfying.

For cost bound $c_{12} = 11$, $a(b_2) = 1$ and $a(b_3) = 1$ (and all others 0) is satisfying.

For cost bound $c_{13} = 12$, there is no satisfying assignment.

# Example: Solving the auction problem

Assumption: Step size 1 and static variable ordering $b_1, b_2, b_3, b_4, b_5$.

For cost bounds from $c_1 = 0$ to $c_9 = 8$, $a(b_1) = 1$ and all others 0 is satisfying.

For cost bound $c_{10} = 9$ and $c_{11} = 10$, $a(b_1) = 1$ and $a(b_5) = 1$ (and all others 0) is satisfying.

For cost bound $c_{12} = 11$, $a(b_2) = 1$ and $a(b_3) = 1$ (and all others 0) is satisfying.

For cost bound $c_{13} = 12$, there is no satisfying assignment.

# Example: Solving the auction problem



Assumption: Step size 1 and static variable ordering $b_1, b_2, b_3, b_4, b_5$.

For cost bounds from $c_1 = 0$ to $c_9 = 8$, $a(b_1) = 1$ and all others 0 is satisfying.

For cost bound $c_{10} = 9$ and $c_{11} = 10$, $a(b_1) = 1$ and $a(b_5) = 1$ (and all others 0) is satisfying.

For cost bound $c_{12} = 11$, $a(b_2) = 1$ and $a(b_3) = 1$ (and all others 0) is satisfying.

For cost bound $c_{13} = 12$, there is no satisfying assignment.

# Branch and Bound

# Branch and Bound: First idea

When solving a COP using a sequence of CSPs, one could use all CSP techniques. However, instead of solving multiple CSPs, one may instead want to integrate the optimization process into the search process.

First idea:

1. Set bound $c = 0$.
2. Use any systematic search technique to find an assignment that satisfies the constraint part.
3. Remember solution in $a$ and global cost in $c$ if global cost $> c$.
4. Return $a$ and $c$ if no further solutions can be found, otherwise continue with next solution at (3).

# Branch and Bound: First idea



When solving a COP using a sequence of CSPs, one could use all CSP techniques. However, instead of solving multiple CSPs, one may instead want to integrate the optimization process into the search process.

First idea:

1. Set bound $c = 0$.
2. Use any systematic search technique to find an assignment that satisfies the constraint part.
3. Remember solution in $a$ and global cost in $c$ if global cost $> c$.
4. Return $a$ and $c$ if no further solutions can be found, otherwise continue with next solution at (3).

# Pruning





Of course, often it is possible to prune the search, even if no inconsistency has been detected yet.

Main idea behind depth-first branch-and-bound (BnB): If the best solution so far is $c$, this is a lower bound for all other possible solutions. So, if a partial solution has led to costs of $x$ for all cost components of fully instantiated variables and the best we can achieve for all other cost components is $y$ with $x + y < c$, then we do not need to continue in this branch.

How can we find out what is the best we can achieve?

# Pruning



Of course, often it is possible to prune the search, even if no inconsistency has been detected yet.

Main idea behind depth-first branch-and-bound (BnB):
If the best solution so far is $c$, this is a lower bound for all other possible solutions. So, if a partial solution has led to costs of $x$ for all cost components of fully instantiated variables and the best we can achieve for all other cost components is $y$ with $x + y < c$, then we do not need to continue in this branch.

How can we find out what is the best we can achieve?

# Pruning



Of course, often it is possible to prune the search, even if no inconsistency has been detected yet.

Main idea behind depth-first branch-and-bound (BnB):
If the best solution so far is $c$, this is a lower bound for all other possible solutions. So, if a partial solution has led to costs of $x$ for all cost components of fully instantiated variables and the best we can achieve for all other cost components is $y$ with $x + y < c$, then we do not need to continue in this branch.

How can we find out what is the best we can achieve?

# Bounding evaluation function



In the following, we will write $\vec{a}_i$ for partial instantiations of the first $i$ variables, assuming a static variable ordering.

## Definition

A bounding evaluation function for a maximizing (minimizing) constraint optimization problem is a function $f$ over partial assignments such that $f(\vec{a}_i) \geq max_a F(a)$ ($f(\vec{a}_i) \leq min_a F(a)$) for all satisfying assignments $a$ that extend $\vec{a}_i$.

Note:

- If $f(\vec{a}_i) < c$ for some already found solution $c$, then $\vec{a}_i$ cannot be extended to a maximal solution.
- $f$ can also be used as a heuristic for choosing a value of the next variable!

# Branch and bound (BnB) algorithm



**BnB**($\mathscr{O}$, $f$):

*Input:* cost network $\mathscr{O}$ and evaluation bounding function $f$
*Output:* an optimal assignment $a'$ (possibly empty) with cost $c'$

```
∀i D'_i ← D_i, i ← 1, c' ← 0, a' ← ∅, a ← ∅
while i ≠ 0
    while 1 ≤ i ≤ n
        remove (v_i ↦ _) from a // remove old assignment to v_i
        x ← SelectValue(i, c')
        if (x = null) D'_i ← D_i // no value for x_i: reset domain
            i ← i − 1 // backtrack
        else a ← a ∪ {v_i ↦ x}
            i ← i + 1 // step forward
    if i = n + 1 // one solution found
        if F(a) > c' // better solution
            a' ← a // remember best solution found so far
            c' ← F(a)
        i ← n // search for next solution
return (a', c')
```

# Branch and bound algorithm: SELECTVALUE



SELECTVALUE($i, c'$):

**while** $D'_i \neq \emptyset$
  select $a^*_i \in D'_i$ such that
    $a^*_i$ = pick one $\arg\max_{a_i \in D'_i} f(a \cup \{v_i \mapsto a_i\})$
  remove $a^*_i$ from $D'_i$
  **if** $a \cup \{v_i \mapsto a^*_i\}$ is consistent **and**
    $f(a \cup \{v_i \mapsto a^*_i\}) > c'$
    **return** $a^*_i$
**return** *null*

# First-choice bounding function



How to come up with a good bounding evaluation function?

In *Operation Research*, one often uses Linear Programming to come up with bounds for Integer Programming Problems.

Let us consider what we can achieve for all soft constraints in isolation subject to the partial assignment we have already. This function is called first-choice (*fc*) bounding function:

$$f_{fc}(\vec{a}_i) = \sum_{F_j \in C_s} \max_{a_{i+1},\ldots,a_n} F_j(\vec{a}_i \cup \{v_{i+1} \mapsto a_{i+1}, \ldots, v_n \mapsto a_n\})$$

How could one improve on that?

- Only allow locally consistent partial assignments.
- Do not consider all soft constraints in isolation, but combine them!

# Example: Auction again



Let us consider BnB with the first-choice bounding function on our auction example:

1. $f_{fc}(\{b_1 \mapsto 1\}) = 8 + (6 + 5 + 2 + 2) = 23$
2. $f_{fc}(\{b_1 \mapsto 1, b_2 \mapsto 0\}) = 8 + (5 + 2 + 2) = 17$
3. $f_{fc}(\{b_1 \mapsto 1, b_2 \mapsto 0, b_3 \mapsto 0\}) = 8 + (2 + 2) = 12$
4. $f_{fc}(\{b_1 \mapsto 1, b_2 \mapsto 0, b_3 \mapsto 0, b_4 \mapsto 0\}) = 8 + (2) = 10$
5. ...

# Russian doll search: Idea



One way to get more accurate bounding functions is to solve subproblems and store the optimal results, reusing them for larger problems.

Solve a sequence of $n$ problems using BnB, where in the $i$th run the last $i$ variables, i.e., $v_{n-i+1}$ up to $v_n$, (and the relevant hard and soft constraints) are considered.

The results of the previous runs can be used:

1. as an initial lower bound,
2. in a heuristic for choosing values, and
3. to generate a more accurate bounding function.

# Improving the evaluation function



- Solve $n$ COPs $\mathcal{O}_i, (i = 1, \ldots, n)$ over the last $i$ variables $v_{n-i+1}, \ldots, v_n$ using BnB and store maximal costs as $c_i^*$.
- In the $(n-i+1)$th run, variables $v_i, \ldots, v_n$ are considered.
- Assume that the variables $v_i, \ldots, v_{i+j}$ are instantiated, denoted by the partial assignment $\vec{a}_j^i$, and that $C_{i,j}$ are all those soft constraints $F$ such that their scopes are included in $\{v_i, \ldots, v_{i+j}\}$.
- Then we use the optimal costs from the $n-i-j$th run to improve on the first-choice function:

$$f(\vec{a}_j^i) = \sum_{F \in C_{i,j}} F(a_i, \ldots, a_j) + c^*_{n-(i+j+1)}.$$

# Conclusion & outlook



- Problems with hard and soft constraints lead to constraint optimization problems
- These are formalized using cost functions and cost networks
- They can be solved using a reduction to a sequence of CSP problems
- More efficiently, one can search for optimal solutions during the backtracking search
- Branch and Bound is the method of choice
- Its pruning power depends on the accuracy of the bounding evaluation function
- Russian doll search can boost its performance
- Further enhancements are possible using constraint inference techniques (such as bucket elimination).

# Literature



Rina Dechter.
Constraint Processing,
Chapter 13, Morgan Kaufmann, 2003