

Constraint Satisfaction Problems

Greedy Local Search

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Constraint solving techniques so far discussed:

- Inference
- Search
- Combinations of inference and search
 - ↔ improve overall performance; nevertheless worst-time complexity is high
- ⇒ approximate solutions, for example, by **greedy local search methods**
- ⇒ in particular of interest, when we look at optimization problems (e.g. traveling salesman problem, minimize violations of so-called **soft constraints**)

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Greedy Local
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Features:

- greedy, hill-climbing traversal of the search space
- in particular, no guarantee to find a solution even if there is one
- search space: states correspond to complete assignment of values to all variables of the constraint network, which are not necessarily solutions of the network
- no systematic search

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SLS ($N, \text{max_tries}, \text{cost}$):

Input: a constraint network N , a number of tries max_tries , a cost function cost

Output: A solution of N or “failure”

repeat max_tries times

 instantiate a complete random assignment $a = (d_1, \dots, d_n)$

repeat

if a is consistent **then return** a

else let Y be the set of assignments that differ from a in exactly one variable-value pair (i.e., change one v_i 's value d_i to a new value d'_i)

$a \leftarrow$ choose an a' from Y with maximal cost improvement

endif

until current assignment cannot be improved

endrepeat

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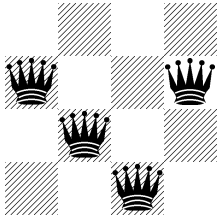
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$$c(a) = 4$$

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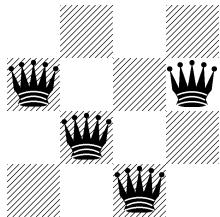
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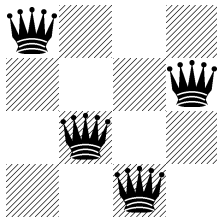
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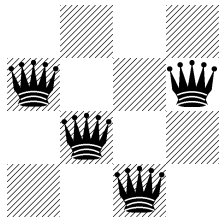
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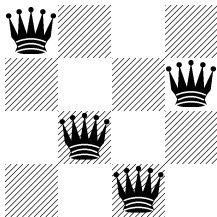
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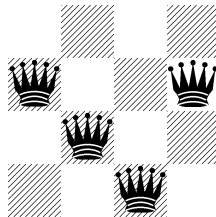
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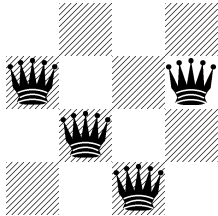
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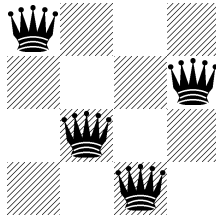
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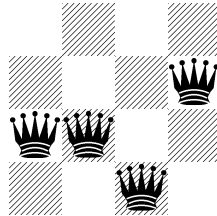
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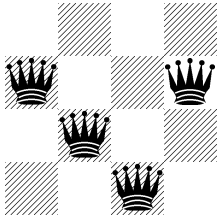
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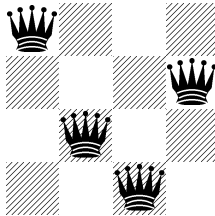
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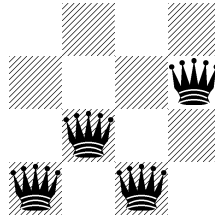
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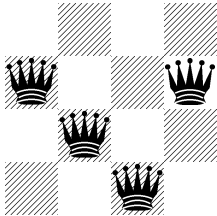
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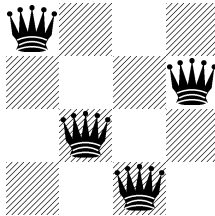
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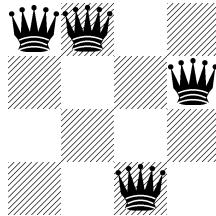
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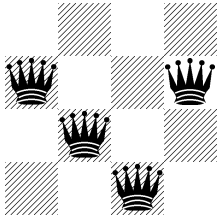
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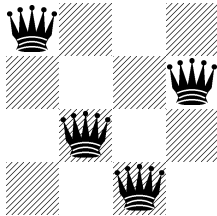
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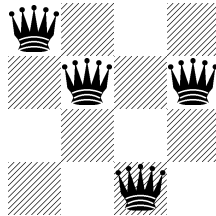
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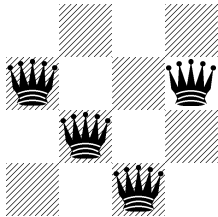
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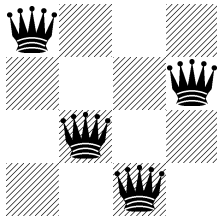
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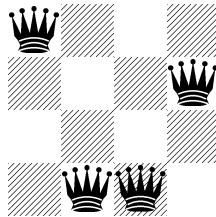
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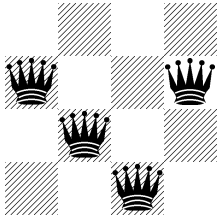
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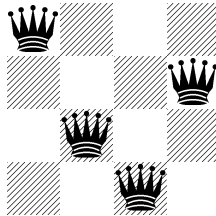
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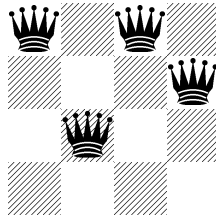
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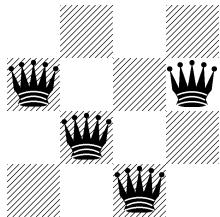
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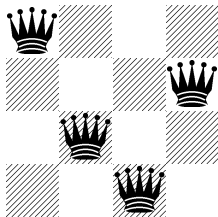
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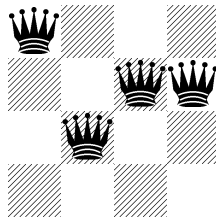
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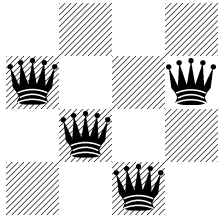
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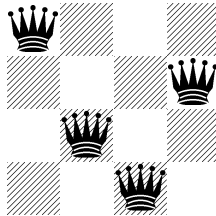
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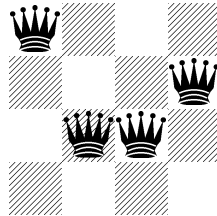
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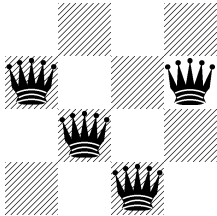
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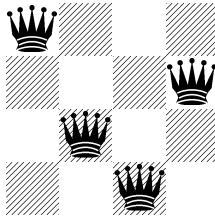
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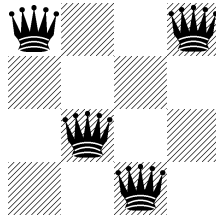
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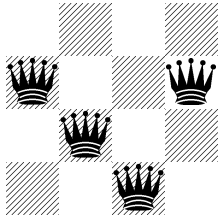
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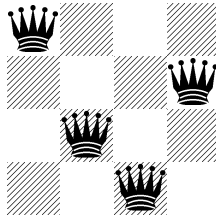
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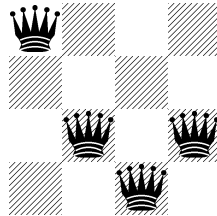
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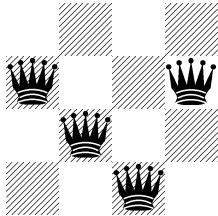
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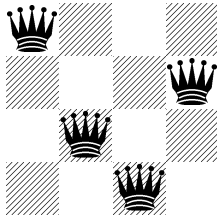
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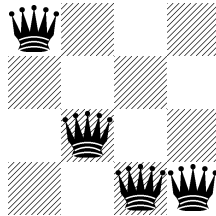
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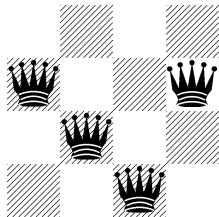
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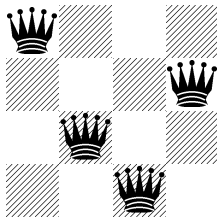
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... is a local minimum,
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In principal, there are two ways for improving the basic SLS-algorithm:

- different strategies for escaping local minima
- other policies for performing local changes

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- **Plateau search:** allow for continuing search by sideways moves that do not improve the assignment

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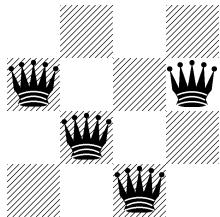
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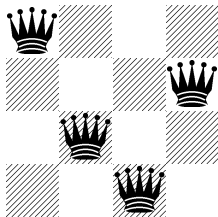
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Example: Plateau search



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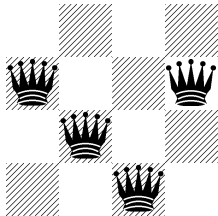
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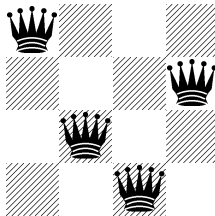
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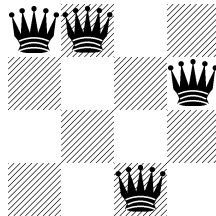
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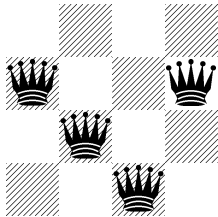
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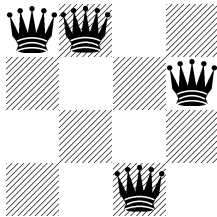
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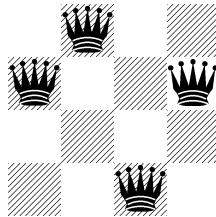
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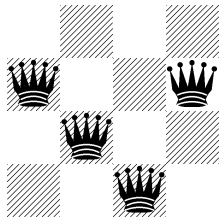
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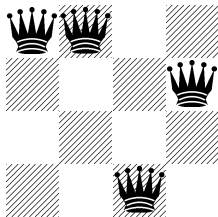
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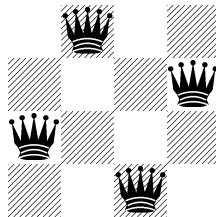
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$$c(a) = 4$$



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$$c(a) = 0$$

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- **Constraint weighting / breakout method:** as a cost measure use a weighted sum of violated constraints; initial weights are changed when no improving move is available.

Idea: if no change reduces the cost of the assignment, increase the weight of those constraints that are violated by the current assignment.

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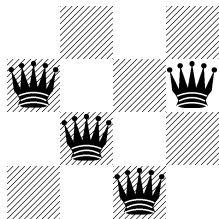
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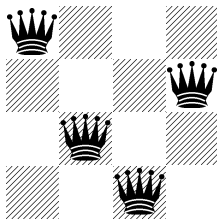
Example: Constraint weighting



$$\begin{array}{lll} w(1,2) = 1 & w(1,3) = 1 & w(1,4) = 1 \\ w(2,3) = 1 & w(2,4) = 1 & w(3,4) = 1 \end{array}$$



$$c(a) = 4$$



$$c(a) = 1$$

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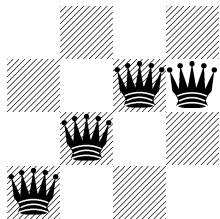
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Example: Constraint weighting



$$\begin{array}{lll} w(1,2) = 1 & w(1,3) = 1 & w(1,4) = 1 \\ w(2,3) = 2 & w(2,4) = 1 & w(3,4) = 1 \end{array}$$



$$c(a) = 5$$

... now the constraint between 2 and 3 is considered more important

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- **Tabu search**: prevent cycling over assignments of the same cost. For this, maintain a list of “forbidden” assignments, called **tabu list** (usually a list of the last n variable-value assignments). The list is updated whenever the assignment changes. Then changes to variable assignments are only allowed w.r.t. to variable-value pairs not in the tabu list.

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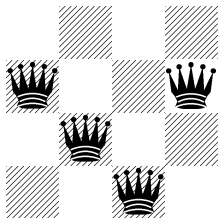
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Example: Tabu search



Tabu list: { (3213) }



$$c(a) = 4$$

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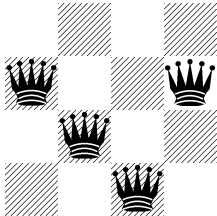
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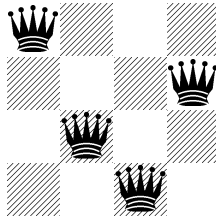
Example: Tabu search



Tabu list: { (3213) (4213) }



$$c(a) = 4$$



$$c(a) = 1$$

... local optimum

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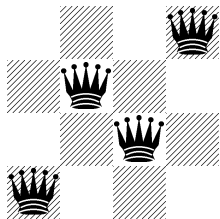
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Example: Tabu search



Tabu list: { (3213) (4213) (1324) }



$$c(a) = 2$$

... restart but no possible improvement

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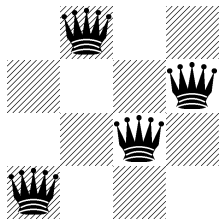
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Example: Tabu search



Tabu list: { (3213) (4213) (1324) (1423) }



$$c(a) = 1$$

...restart and so on

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Random walk strategy:

- combines random walk search with a greedy approach (bias towards assignments that satisfy more constraints)
- instead of making greedy moves in each step, sometimes perform a random walk step
- for example, start from a random assignment. If the assignment is not a solution, select randomly an unsatisfied constraint and change the value of one of the variables participating in the constraint.

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WalkSAT:

- initially formulated for SAT solving (by Selman, Kautz, & Cohen: WALKSAT/SKC)
- turns out to be very successful (in empirical studies)
- based on a two-stage process for selecting variables: in each step select first a constraint violated by the current assignment; second make a random choice between
 - a) changing the value of one of the variables in the violated constraint;
 - b) minimizing in a greedy way the **break value**, i.e., the number of new constraints that become inconsistent by changing a value

The choice between (a) and (b) is controlled by a parameter p (probability for (a))

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WalkSAT ($N, \text{max_flips}, \text{max_tries}$):

Input: a constraint network N , numbers max_flips (flips) and max_tries (tries)

Output: “true” and a solution of N , or
“failure” and some inconsistent best assignment

$a' \leftarrow$ a complete random assignment

repeat max_tries times

$a = (d_1, \dots, d_n) \leftarrow$ a complete random assignment

repeat max_flips times

if a is consistent **then return** “true” and a

else select a violated constraint C with scope s

with probability p : choose an arbitrary variable-value pair (v_i, d') ,

$v_i \in s, d_i \neq d'$

else (with probability $1 - p$): choose a variable-value pair (v_i, d') ,

$v_i \in s, d_i \neq d'$, that maximizes the number of satisfied

constraints when v_i 's value in a is changed to d'

$a \leftarrow (a \text{ with } v_i \mapsto d')$

endif

endrepeat

compare a with a' and retain the better one as a'

endrepeat

return “failure” and a'



Simulated Annealing:

- **Idea:** over time decrease the probability of doing a random move over one that maximally decreases costs. Metaphorically speaking, by decreasing the probability of random moves, we “freeze” the search space.
- At each step, select a variable-value pair and compute the change of the cost function, δ , when the value of the variable is changed to the selected value. Change the value if δ is not negative (i.e., costs do not increase). Otherwise, we perform the change with probability $e^{\delta/T}$ where T is the temperature parameter ($T \geq 0$).
- The temperature T is decreased over time (schedule): more random moves are allowed at the beginning and less such moves at the end.

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SimulatedAnnealing ($N, schedule$):

Input: a constraint network N , a cost function $cost$ and a schedule $schedule$ mapping time to temperature

Output: A solution candidate to N

instantiate a complete random assignment $a = (d_1, \dots, d_n)$

for $t = 1, 2, 3, \dots$

$T \leftarrow schedule(t)$

if $T = 0$ **then return** a

$a' \leftarrow$ a complete random assignment

$\delta \leftarrow cost(a) - cost(a')$

if $\delta > 0$ **then** $a \leftarrow a'$

else $a \leftarrow a'$ with probability $e^{\delta/T}$

endfor

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Stochastic local search methods can be applied to many combinatorial problems (such as CSP). An abstract characterization of these methods is as follows:

Given a combinatorial problem X a **stochastic local search algorithm** for solving instances x of X is specified by:

- the **search space** S_x of x (elements are referred to as **locations, positions, or configurations**)
- a set of **feasible solutions** $S_x^* \subseteq S_x$
- a **neighborhood relation** N_x on S_x representing which positions can be reached from another position in one search step.
- a (finite) set M_x of **memory states** (representing, e.g., previously visited states)
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- an **initialization method** $init_x$ that specifies the initialization of the search: the result is a probability distribution over $S_x \times M_x$
- a **step function** $step_x: S_x \times M_x \rightarrow \Pi(S_x \times M_x)$, assigning to each position and memory state a probability distribution over the neighboring positions and memory states
- a **termination function** $terminate_x: S_x \times M_x \rightarrow \Pi(\{0, 1\})$, providing a probability distribution of the probability by which the search is terminated when the search has reached a certain position and memory state.

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Given a constraint network N :

- Search space: the set of all complete assignments of N
- Solutions: the consistent assignments (solutions) of N
- Neighborhood: typically **1-exchange neighborhood**, i.e., two positions are considered neighbor if they differ at most in the assignment of a single variable (in SAT: **1-flip neighborhood**)
- Initialization: mostly random assignment (with uniform distribution)
- Step function: this is where most algorithms we saw differ i.e., these algorithm use different **heuristics** for selecting the next step
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Definition (Hoos)

A stochastic local search algorithm is **probabilistically approximately complete** (PAC) if on all solvable instances the probability that the algorithm finds a solution of the instance within time t goes to 1 as t goes to ∞ .

Notice:

Assume that the neighborhood relation is connected (each position is reachable from each other position) and all search steps have a probability > 0 . Then purely random walk (no heuristic guidance) has the PAC property.

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Most heuristics use an **evaluation function** g mapping assignments to non-negative real numbers such that the global minima of g correspond to solutions.

In the CSP context, g is most of the times simply chosen such that the number of violated constraints are counted (see previous slides).

Most popular heuristics is the **min-conflict heuristic**: randomly select new value for a variable randomly selected from the variables in some unsatisfied constraint under the current assignment such that the number of unsatisfied constraints is minimized (see SLS).

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To escape local minima, **random walk steps** are performed (this often guarantees PAC property), in particular, the random walk probability (*noise setting*) must be > 0 .

If the random walk steps modify the assignment of variable in an unsatisfied constraint, we say that the random walks are **conflict-directed**.

Random walks can be combined with restarts ... (see WalkSAT).

Does this pay off? PAC-property when the number of restarts (`max_tries`) is fixed? Experimental results crucially depend on instances and the settings of the parameters used.

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SLS-algorithms can also be combined with inference methods. For example, apply SLS only after preprocessing a given CSP instance with some consistency-enforcing algorithm.

Idea: Can we improve SLS by looking at equivalent but more explicit constraint networks?

Note:

- there are classes of problems, e.g., 3SAT problems, which can easily be solved by a systematic backtracking algorithm, but are hard to be solved via SLS
- consistency-enforcing algorithms can change the costs associated to an arc in the constraint graph drastically: assignments near to a solution (in terms of costs) may be very far from a solution after applying inference methods

Example:

- Local search on cycle cutsets

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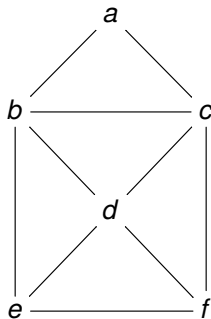
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Cycle-cutset:an example



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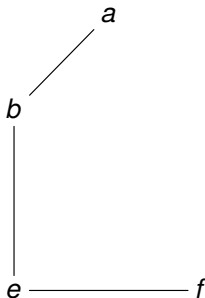
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Cycle-cutset:an example



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Idea for a hybrid algorithm:

- 1 Determine a cycle cutset
- 2 Find some assignment for the cutset variables
- 3 Find assignment for the tree variables that minimizes costs, given the assignment to the cutset variables
- 4 Do stochastic local search by varying the cutset variables only
- 5 Continue with step 3 if there was some improvement
- 6 Otherwise stop

Usually outperforms pure SLS, provided the cutset is small ($\leq 30\%$).

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SLS algorithms ...

- are anytime: the longer the run, the better the solution they produce (in terms of a cost function counting violated constraints)
- terminate at local minima
- cannot be used to prove inconsistency of CSP instances

However, WalkSAT can be shown to find a satisfying assignment with probability approaching 1, provided the procedure can run long enough (exponentially long) and provided such an assignment exists.

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