Constraint Satisfaction Problems

Constraint Networks

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1 Constraint Networks

- Constraint networks
- Solution
- Normalized Constraint Networks
- Deduction

Constraint networks

Definition

A constraint network is a triple

$$N = \langle V, \mathrm{dom}, C \rangle$$

where:

- V is a non-empty and finite set of variables;
- dom is a function that assigns to each variable $v \in V$ a non-empty set $\mathrm{dom}(v)$ ($\mathrm{dom}(v)$ is called the domain of v, elements of $\mathrm{dom}(v)$ are called values);
- C is a set of relations over variables of V (called constraints), i.e., each constraint is a relation $R_{x_1,\dots,x_m}$ over some scheme $S = (x_1, \dots, x_m)$ of variables in V.

The set of constraint schemes $\{S_1, \dots S_t\}$ is called network scheme.

Constraint networks

If we assume some ordering of the variables in V, we can write networks more compactly:

Definition

A constraint network is a triple $N = \langle V, D, C \rangle$ where:

- $V = \{v_1, \dots, v_n\}$ is a non-empty and finite ordered set of variables (assume order $(v_1, \dots, v_n)$);
- $D = (D_1, \dots, D_n)$ is a sequence of domains for V (with D_i the domain of variable v_i and $D_N = D_1 \times \cdots \times D_n$ is the domain of N);
- C is a set of constraints (x, R) where $x = (v_{i_1}, \dots, v_{i_m})$ is a scheme of variables in V and $R \subseteq D_{i_1} \times \cdots \times D_{i_m}$.

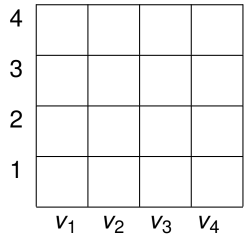
Notice: any ordering of the variables suffices. Constraint network that differ only in the variable ordering are considered equal.

Constraint networks

- Note that we consider finitely many variables only, but (e.g., for theoretical studies) this could be relaxed.
- In the 2nd definition we assumed that the relations of the constraint are embedded in the domain of the network: $R \subseteq D_{i_1} \times \cdots \times D_{i_m}$. Such networks are called embedded networks (see Bessiere, 2006).
- The definition does not require that constraint relations are given explicitly (in extension, i.e. by a set of its tuples; table constraint). A constraint relation R could be specified by any Boolean function f_R: a tuple satisfies the constraint R iff f_R applied on the tuple gives 1.
- If not stated otherwise, we will always assume that the domains of the variables are given in extension.

Example: 4-queens problem

The 4-queens problem can be represented as a constraint network. For example, consider variables $v_1, \ldots, v_4$ (each associated to a column of the 4×4-chess board). Each variable v_i has as its domain $D_i = \{1, \ldots, 4\}$ (conceived of as the row positions of a queen in column i).



Define then binary constraints (thus encoding "non-attacking queen positions"):

$$R_{v_1,v_2} := \{(1,3),(1,4),(2,4),(3,1),(4,1),(4,2)\}$$

$$R_{v_1,v_3} := \{(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\}$$

...

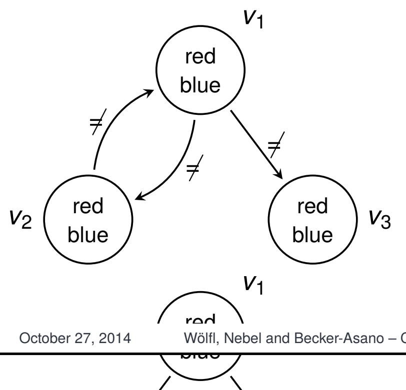
Example: Graph colorability

k-Colorability of a graph G can be represented as a constraint network of the following form:

$$V = \{v_i : v_i \text{ is a vertex in } G\}$$

$$D_i = \{1, \ldots, k\} \quad (v_i \in V)$$

$$C = \{((v_i, v_j), \neq) : \{v_i, v_j\} \text{ is an edge of } G\}$$



Binary constraint networks can be represented by a directed labeled graph (or even: by an undirected graph if all constraints are

Solution of a constraint network

Definition

A solution of a constraint network $N = \langle V, D, C \rangle$ is a (variable) assignment

$$a\colon V \to \bigcup_{i\colon v_i \in V} D_i$$

such that:

(a) $a(v_i) \in D_i$, for each $v_i \in V$,

(b) $(a(x_1), \ldots, a(x_m)) \in R$ for each constraint $R_{x_1,\ldots,x_m}$ in C.

N is called satisfiable if N has a solution.

sol(N) denotes the set of all solutions of N. sol(N) can also be written as:

$$\text{sol}(N) = \{(d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n : \text{the assignment } v_1 \mapsto d_1, \ldots, v_n \mapsto d_n \text{ defines a solution}\}$$

Instantiation, partial solution

Let $N = \langle V, D, C \rangle$ be a constraint network.

Definition

(a) An instantiation of a subset $V' \subseteq V$ is an assignment $a : V' \to \bigcup_{i:\, v_i \in V'} D_i$ with $a(v_i) \in D_i$.

(b) An instantiation a of V' is called partial solution if a satisfies each constraint R_S in C with $S \subseteq V'$.
In this case a is called locally consistent.

(c) Shortcut notation: for an instantiation a of $V' = \{x_1, \ldots, x_m\}$ and constraint R_S with scope $S \subseteq V'$, set

$$a[S] := (a(x_1), \ldots, a(x_m)).$$

Hence, a solution is an instantiation of all variables in V that is locally consistent.

Instantiation, solution

Note:

(a) An instantiation of $V' \subseteq V$, a, is a partial solution (locally consistent) iff

$$a[S] \in R, \quad \text{for each constraint } R \text{ with scope } S \subseteq V'.$$

(b) Not every partial solution is part of a (full) solution, i.e., there may be partial solutions of a constraint network that cannot be extended to a solution. For the 4-queens problem, for example:

4	q			
3			q	
2				
1		q		
	v_1	v_2	v_3	v_4

Nogoods

Let $N = \langle V, D, C \rangle$ be a constraint network.

Definition

An instantiation a' of subset $V' \subseteq V$ is called a nogood (of N) if a' cannot be extended to a (full) solution of N, i.e., there exists no solution $a\colon V \to \bigcup_i D_i$ such that $a|_{V'} = a'$.

Instantiations that are no nogoods are sometimes called consistent or globally consistent (to emphasize the difference to locally consistent assignments).

Later, we will also introduce the notion of globally consistent networks.

Normalized constraint network

Let $N = \langle V, D, C \rangle$ be a constraint network.
Due to our definition it is possible that C contains constraints

$$R_{v_{i_1},\ldots,v_{i_k}} \quad \text{and} \quad S_{v_{j_1},\ldots,v_{j_k}}$$

where $(j_1, \ldots, j_k)$ is just a permutation of $(i_1, \ldots, i_k)$.
Without changing the set of solutions, we can simplify the network by deleting $S_{v_{j_1},\ldots,v_{j_k}}$ from C and rewriting $R_{v_{i_1},\ldots,v_{i_k}}$ as follows:

$$R_{v_{i_1},\ldots,v_{i_k}} \leftarrow R_{v_{i_1},\ldots,v_{i_k}} \cap \pi_{v_{i_1},\ldots,v_{i_k}}(S_{v_{j_1},\ldots,v_{j_k}}).$$

Given a fixed order on the set of variables V, we can systematically delete-and-refine constraints. This results in a constraint network that contains at most one constraint for each subset of variables. Such a network is called a normalized constraint network.

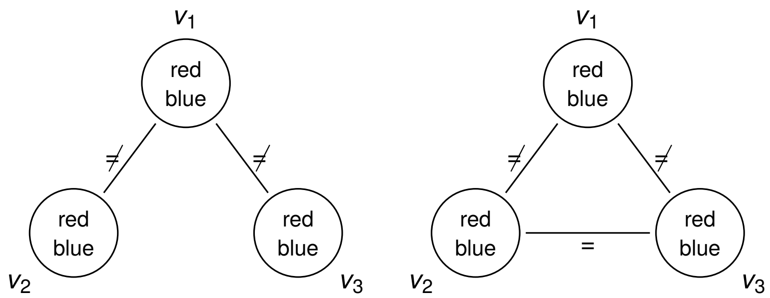
Equivalence

Let N and N' be constraint networks on the same set of variables and on the same domains for each variable.

Definition

N and N' are called equivalent if they have the same set of solutions.

Example:



Tightness

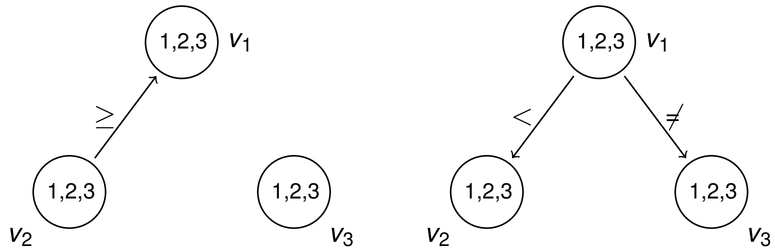
Let N and N' be (normalized) constraint networks on the same set of variables and on the same domains for each variable.

Definition

N is as tight as N' if for each constraint R_S of N,

(a) N' has no constraint with the same scope as R_S, or

(b) $R \subseteq \pi_S(R'_{S'})$, where $R'_{S'}$ is the constraint of N' with the same scope as R_S.



- Clearly, if N' is as tight as N, then $\text{sol}(N') \subseteq \text{sol}(N)$.

Intersection of networks

Definition

The intersection of N and N', $N \cap N'$, is the network defined by intersecting for each scope the constraints $R_S \in C$ and $R'_{S'} \in C'$ with the same scope, i.e., modulo a suitable permutation of the constraint schemes,

$$R''_S := R_S \cap R'_S.$$

If for a scope S only one of the networks contains a constraint, then we set:

$$R''_S := R_S \quad (\text{or } := R'_S, \text{ resp.})$$

Lemma

If N and N' are equivalent networks, then $N \cap N'$ is equivalent to both networks and as tight as both networks.

2 Constraint Networks and Graphs

- Primal Constraint Graphs
- Dual Constraint Graph
- Constraint Hypergraph

Primal constraint graphs

Let $N = \langle V, D, C \rangle$ be a (normalized) constraint network.

Definition

The primal constraint graph of a network $N = \langle V, D, C \rangle$ is the undirected graph

$$G_N := \langle V, E_N \rangle$$

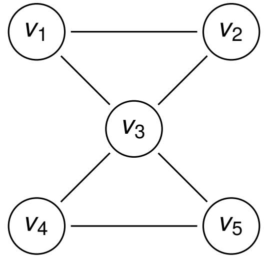
where

$\{u, v\} \in E_N \iff \{u, v\}$ is a subset of the scope of some constraint in N.

Primal constraint graph: Example

Consider a constraint network with variables $v_1, \ldots, v_5$ and two ternary constraints R_{v_1,v_2,v_3} and S_{v_3,v_4,v_5}.

Then the primal constraint graph of the network has the form:



Absence of an edge between two variables/nodes means that there is no explicit constraint in which both variables participate.

Dual constraint graphs

Definition

The dual constraint graph of a constraint network $N = \langle V, D, C \rangle$ is the labeled graph

$$D_N := \langle V', E_N, l \rangle$$

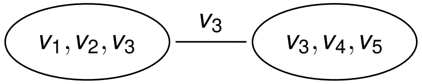
with

$$X \in V' \iff X \text{ is the scope of some constraint in } N$$

$$\{X, Y\} \in E_N \iff X \cap Y \neq \emptyset$$

$$l : E_N \to 2^V, \quad \{X, Y\} \mapsto X \cap Y$$

In the example above, the dual constraint graph is:



Constraint hypergraph

Definition

The constraint hypergraph of a constraint network $N = \langle V, D, C \rangle$ is the hypergraph

$$H_N := \langle V, E_N \rangle$$

with

$$X \in E_N \iff X \text{ is the scope of some constraint in } N.$$

In the example above (constraint network with variables $v_1, \ldots, v_5$ and two ternary constraints R_{v_1,v_2,v_3} and S_{v_3,v_4,v_5}) the hyperedges of the constraint hypergraph are:

$$E_N = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}\}.$$

3 Solving Constraint Networks

Simple solution strategy: Backtracking search

Backtracking: search systematically for locally consistent partial instantiations in a depth-first manner:

- forward phase: extend the current partial solution by assigning a consistent value to some new variable (if possible)
- backward phase: if no consistent instantiation for the current variable exists, return to the previous variable.

Backtracking algorithm

Backtracking(N, a):

Input: a constraint network $N = \langle V, D, C \rangle$ and
a partial assignment a of N
(e.g., the empty instantiation $a = \{\,\}$)
Output: a solution of N or “inconsistent”

if a is not locally consistent with N:
    return “inconsistent”
if a is defined for all variables in V:
    return a
select some variable v_i for which a is not defined
for each value x from D_i:
    $a' := a \cup \{v_i \mapsto x\}$
    $a'' \leftarrow$ Backtracking(N, a')
    if a'' is not “inconsistent”:
        return a''
return “inconsistent”

Literature

Rina Dechter.
Constraint Processing,
Chapter 2, Morgan Kaufmann, 2003