Constraint Satisfaction Problems

Introduction

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

Stefan Wölfl, **Christian Becker-Asano**, and **Bernhard Nebel**

October 20, 2014

Introduction

Constraints

What is a constraint?

1 a: the act of constraining **b:** the state of being checked, restricted, or compelled to avoid or perform some action . . .
c: a constraining condition, agency, or force . . .
2 a: repression of one's own feelings, behavior, or actions **b:** a sense of being constrained . . .

(from *Merriam-Webster's Online Dictionary*)

Usage

- In programming languages, constraints are often used to restrict the domains of variables.
- In databases, constraints can be used to specify integrity conditions.
- In mathematics, a constraint is a requirement on solutions of optimization problems.

Examples

Examples:

- Latin squares
- Eight queens problem
- Sudoku
- Map coloring problem
- Boolean satisfiability

Latin square

Problem:

- How can one fill an $n \times n$ table with n different symbols
- … such that each symbol occurs exactly once in each row and in each column?

$$\begin{bmatrix}1\end{bmatrix} \quad \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 4 & 3\\ 2 & 3 & 1 & 4\\ 3 & 4 & 2 & 1\\ 4 & 1 & 3 & 2\end{bmatrix}$$

There are 56 different *reduced* Latin squares of size 5, 9408 squares of size 6, 16.942.080 squares of size 7, 535.281.401.856 squares of size 8, …

Latin square

Problem:

- How can one fill an $n \times n$ table with n different symbols
- … such that each symbol occurs exactly once in each row and in each column?

$$\begin{bmatrix}1\end{bmatrix} \quad \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\\ 3 & 4 & 1 & 2\\ 4 & 1 & 2 & 3\end{bmatrix}$$

There are 56 different *reduced* Latin squares of size 5, 9408 squares of size 6, 16.942.080 squares of size 7, 535.281.401.856 squares of size 8, …

Sudoku

Problem:

- Fill a partially completed 9×9 grid such that
- … each row, each column, and each of the nine 3×3 grids contains the numbers from 1 to 9.

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

Sudoku

Problem:

- Fill a partially completed 9×9 grid such that
- … each row, each column, and each of the nine 3×3 grids contains the numbers from 1 to 9.

2	5	8	7	3	6	9	4	1
6	1	9	8	2	4	3	5	7
4	3	7	9	1	5	2	6	8
3	9	5	2	7	1	4	8	6
7	6	2	4	9	8	1	3	5
8	4	1	6	5	3	7	2	9
1	8	4	3	6	9	5	7	2
5	7	6	1	4	2	8	9	3
9	2	3	5	8	7	6	1	4

Eight queens puzzle

Problem:

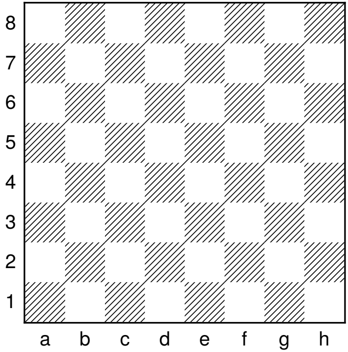
- How can one put 8 queens on a standard chess board (8×8-board)
- ...such that no queen can attack any other queen?

Solutions:

- The puzzle has 12 unique solutions (up to rotations and reflections)

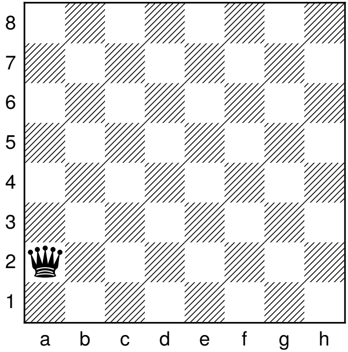
- Old problem proposed in 1848.
- Various variants
  - knights (instead of queens)
  - 3D
  - n queens on an $n \times n$-board

A solution ...



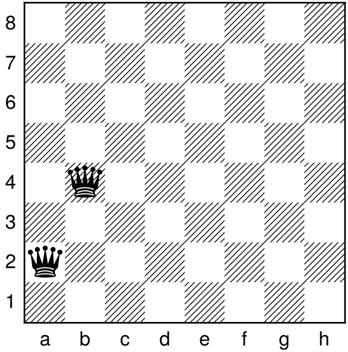
A solution of the 8-queens problem

A solution …



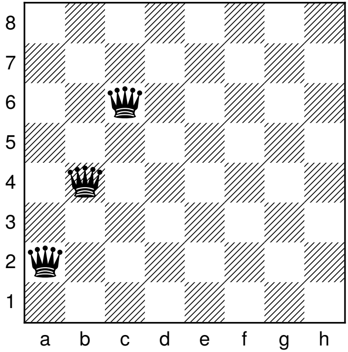
A solution of the 8-queens problem

A solution …



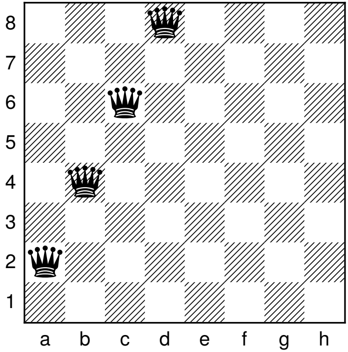
A solution of the 8-queens problem

A solution . . .



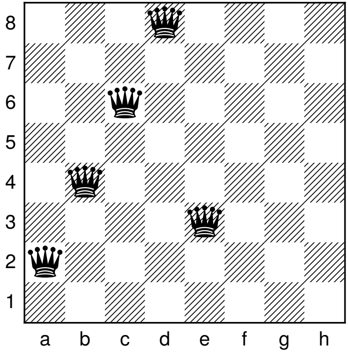
A solution of the 8-queens problem

A solution ...



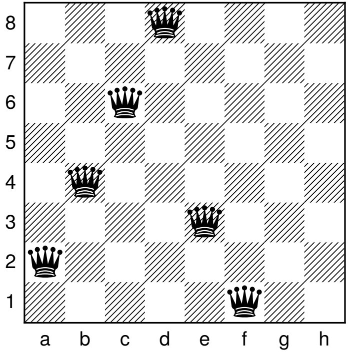
A solution of the 8-queens problem

A solution . . .



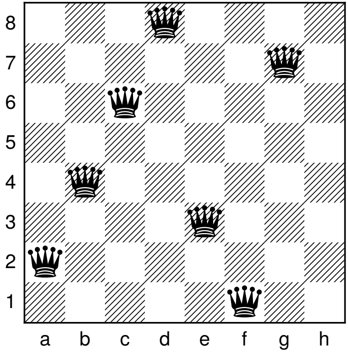
A solution of the 8-queens problem

A solution …



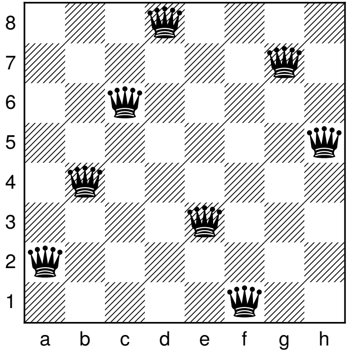
A solution of the 8-queens problem

A solution ...



A solution of the 8-queens problem

A solution ...



A solution of the 8-queens problem

Constraint Satisfaction Problem (CSP)

Definition

A constraint network is defined by:

- a finite set of variables
- a (finite) domain of values for each variable
- a finite set of constraints (i.e., binary, ternary, … relations defined between the variables)

Problem

Is there a solution of the network, i.e., an assignment of values to the variables such that all constraints are satisfied?

Constraint Satisfaction Problem (CSP)

Definition

A constraint network is defined by:

- a finite set of variables
- a (finite) domain of values for each variable
- a finite set of constraints (i.e., binary, ternary, . . . relations defined between the variables)

Problem

Is there a solution of the network, i.e., an assignment of values to the variables such that all constraints are satisfied?

k-Colorability

Problem:

- Can one color the nodes of a given graph with k colors
- … such that all nodes connected by an edge have different colors?

Reformulated as a constraint network:

- Variables: the nodes of the graph
- Domains: "colors" $\{1,\ldots,k\}$ for each variable
- Constraints: nodes connected by an edge must have different values

This constraint network has a particular restricted form:

- only binary constraints
- domains are finite

k-Colorability

Problem:

- Can one color the nodes of a given graph with k colors
- … such that all nodes connected by an edge have different colors?

Reformulated as a constraint network:

- Variables: the nodes of the graph
- Domains: "colors" $\{1,\ldots,k\}$ for each variable
- Constraints: nodes connected by an edge must have different values

This constraint network has a particular restricted form:

- only binary constraints
- domains are finite

k-Colorability

Problem:

- Can one color the nodes of a given graph with k colors
- … such that all nodes connected by an edge have different colors?

Reformulated as a constraint network:

- Variables: the nodes of the graph
- Domains: "colors" $\{1,\ldots,k\}$ for each variable
- Constraints: nodes connected by an edge must have different values

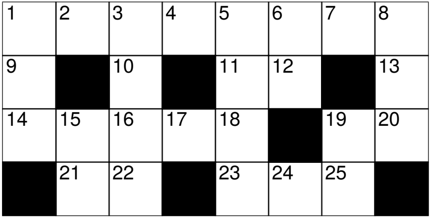
This constraint network has a particular restricted form:

- only binary constraints
- domains are finite

Crossword puzzle

Problem instance:

- Variables: empty squares in a crossword puzzle;
- Domains: letters $\{A, B, C, \ldots, Z\}$ for each variable;
- Constraints: relations defined by a given set of words that need (or are allowed) to occur in the completed puzzle.



Crossword puzzle

UNI FREIBURG

Problem instance:

- Variables: empty squares in a crossword puzzle;
- Domains: letters $\{A, B, C, \ldots, Z\}$ for each variable;
- Constraints: relations defined by a given set of words that need (or are allowed) to occur in the completed puzzle.

Z	U	H	O	E	R	E	R
A	■	O	■	I	E	■	A
R	O	L	L	E	■	I	M
■	N	Z	■	R	O	T	■

Boolean satisfiability

SAT

Given a propositional logic formula in CNF, is the formula satisfiable?

As a constraint satisfaction problem:

Problem instance (Boolean constraint network):

- Variables: (propositional) variables;
- Domains: truth values $\{0, 1\}$ for each variable;
- Constraints: defined by a clause in the formula.

Example: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee x_4)$

Traveling salesperson problem

Traveling salesperson problem (TSP):

Given a set of n cities and distances c_{ij} between city i and city j, find the shortest route that visits all cities and finishes in the starting city.

TSP is not a constraint satisfaction problem, but a constraint optimization problem

…

Vehicle routing problem

Vehicle routing problem (VRP):

Given a set of goods that to need to be delivered from a central depot to costumers; and given a fleet of trucks that can transport the goods: find an assignment of routes to the trucks minimizes the total route cost.

Dozens of variants: Capacitated Vehicle Routing Problem (CVRP), ... with Pickup and Delivery (VRPPD), ... with time windows (CRPTW), ...

Vehicle routing problem

Vehicle routing problem (VRP):

Given a set of goods that to need to be delivered from a central depot to costumers; and given a fleet of trucks that can transport the goods: find an assignment of routes to the trucks minimizes the total route cost.

Dozens of variants: Capacitated Vehicle Routing Problem (CVRP), ... with Pickup and Delivery (VRPPD), ... with time windows (CRPTW), ...

Real world applications

In practice, not only constraint satisfaction, but constraint optimization is required.

Seminar topic assignment

- Given n students who want to participate in a seminar; m topics are available to be worked on by students; each topic can be worked on by at most one student, and each student has preferences which topics s/he would like to work on;
- … how to assign topics to students?

Real world applications

CSP/COP techniques can be used in

- *civil engineering* (design of power plants, water and energy supply, transportation and traffic infrastructure)
- *mechanical engineering* (design of machines, robots, vehicles)
- digital circuit *verification*
- automated timetabling
- air traffic control
- finance

Computational complexity

Theorem

It is NP-hard to decide solvability of CSPs.

Since *k*-colorability (SAT, 3SAT) is NP-complete, solvability of CSPs in general must be NP-hard.

Question: Is CSP solvability *in* NP?

Computational complexity

Theorem

It is NP-hard to decide solvability of CSPs.

Since k-colorability (SAT, 3SAT) is NP-complete, solvability of CSPs in general must be NP-hard.

Question: Is CSP solvability *in* NP?

Computational complexity

Theorem

It is NP-hard to decide solvability of CSPs.

Since *k*-colorability (SAT, 3SAT) is NP-complete, solvability of CSPs in general must be NP-hard.

Question: Is CSP solvability *in* NP?

Solving CSPs

- Enumeration of all assignments and testing

⇝ … too costly

- Backtracking search

⇝ numerous different strategies, often “dead” search paths are explored extensively

- Constraint propagation: elimination of obviously impossible values
- Interleaving backtracking and constraint propagation: constraint propagation at each generated search node
- Many other search methods, e. g., local/stochastic search, etc.

Solving CSPs

- Enumeration of all assignments and testing

⇝ … too costly

- Backtracking search

⇝ numerous different strategies, often “dead” search paths are explored extensively

- Constraint propagation: elimination of obviously impossible values
- Interleaving backtracking and constraint propagation: constraint propagation at each generated search node
- Many other search methods, e. g., local/stochastic search, etc.

Solving CSPs

- Enumeration of all assignments and testing

⇝ … too costly

- Backtracking search

⇝ numerous different strategies, often “dead” search paths are explored extensively

- Constraint propagation: elimination of obviously impossible values
- Interleaving backtracking and constraint propagation: constraint propagation at each generated search node
- Many other search methods, e. g., local/stochastic search, etc.

Solving CSPs

- Enumeration of all assignments and testing

⇝ … too costly

- Backtracking search

⇝ numerous different strategies, often “dead” search paths are explored extensively

- Constraint propagation: elimination of obviously impossible values
- Interleaving backtracking and constraint propagation: constraint propagation at each generated search node
- Many other search methods, e. g., local/stochastic search, etc.

Solving CSPs

- Enumeration of all assignments and testing

⇝ … too costly

- Backtracking search

⇝ numerous different strategies, often "dead" search paths are explored extensively

- Constraint propagation: elimination of obviously impossible values
- Interleaving backtracking and constraint propagation: constraint propagation at each generated search node
- Many other search methods, e. g., local/stochastic search, etc.

Contents I

- *Introduction and mathematical background*
  - Sets, relations, graphs
  - Constraint networks and satisfiability
  - Binary constraint networks
  - Simple solution methods (backtracking, etc.)
- *Inference-based methods*
  - Arc and path consistency
  - k-consistency and global consistency
- *Search methods*
  - Backtracking
  - Backjumping
  - Comparing different methods
  - Stochastic local search

Contents I

- *Introduction and mathematical background*
  - Sets, relations, graphs
  - Constraint networks and satisfiability
  - Binary constraint networks
  - Simple solution methods (backtracking, etc.)
- *Inference-based methods*
  - Arc and path consistency
  - *k*-consistency and global consistency
- *Search methods*
  - Backtracking
  - Backjumping
  - Comparing different methods
  - Stochastic local search

Contents I

- *Introduction and mathematical background*
  - Sets, relations, graphs
  - Constraint networks and satisfiability
  - Binary constraint networks
  - Simple solution methods (backtracking, etc.)
- *Inference-based methods*
  - Arc and path consistency
  - k-consistency and global consistency
- *Search methods*
  - Backtracking
  - Backjumping
  - Comparing different methods
  - Stochastic local search

Contents II

- *Global constraints*
- *Constraint optimization*
- *Selected advanced topics*
  - Expressiveness vs complexity of constraint formalisms
  - Qualitative constraint networks

Contents II

- *Global constraints*
- *Constraint optimization*
- *Selected advanced topics*
  - Expressiveness vs complexity of constraint formalisms
  - Qualitative constraint networks

Contents II

- *Global constraints*
- *Constraint optimization*
- *Selected advanced topics*
  - Expressiveness vs complexity of constraint formalisms
  - Qualitative constraint networks

Organization

Lectures: Where, when, web page

Where

Bld. 51, Room SR 00 006

When

Monday, 10:15–12:00

Wednesday, 10:15–11:00 (+ exercises: 11:15–12:00)

No lectures

- 24-12-2014 – 06-01-2015 (Christmas break)

Web Page

`http://www.informatik.uni-freiburg.de/~ki/teaching/ws1415/csp/`

Lecturers

Prof. Bernhard Nebel

Bld. 52, Room 00-029

Phone: 0761/203-8221

Email: nebel@informatik.uni-freiburg.de

Dr. Christian Becker-Asano

Bld. 52, Room 00-042

Phone: 0761/203-8251

Email: basano@informatik.uni-freiburg.de

Dr. Stefan Wölfl

Bld. 52, Room 00-043

Phone: 0761/203-8228

Email: woelfl@informatik.uni-freiburg.de

Exercises

Where

Bld. 51, Room SR 00 006

When

Wednesday, 11:15–12:00

Course prerequisites & goals

Goals

- Acquiring skills in constraint processing
- Understanding the principles behind different solving techniques
- Being able to read and understand research literature in the area of constraint satisfaction
- Being able to complete a project (thesis) in this research area

Prerequisites

- Basic knowledge in the area of AI
- Basic knowledge in formal logic
- Basic knowledge in theoretical computer science

Exercise assignments

- handed out on Wednesdays
- due on Wednesday in the following week (before the lecture)
- may be solved in groups of **two** students
- 50 % of reachable points are required for exam admission

Programming project

Implement a CSP solver . . .

- Implementation tasks are specified on a regular basis (depending on the progress of the lecture)
- Programming language
- Implementation should compile on a standard Linux computer (Ubuntu 13.08)
- We provide git repositories for source code
- Working solver is prerequisite for exam admission
- We will do a competition between solvers at the end of the lecture

Examination

Credit points

- 6 ECTS points

Exams

- (Oral or written) exam in February/March 2015

Projects and theses

Topics of theses resulting from this lecture:

- *Räumliche und zeitliche Constraints in beschreibungslogischen Wissensbasen*
- *Tableaux-Verfahren zur Lösung qualitativer CSPs*
- *Revisionsoperationen auf qualitativen Constraintnetzen*
- *Berechnung handhabbarer Klassen für qualitative räumliche Formalismen*
- *Fast procedures for the combination of qualitative constraint calculi*

Projects and theses

Topics of projects related to this lecture:

- *Ein Schwierigkeitsmaß für Sudoku-Puzzles*
- *Empirische Analyse von Konsistenz- und Suchalgorithmen*
- *Umsetzung eines CSP-Methoden-basierten Timetabling Algorithmus*

Acknowledgements

Lecture is based on slidesets of previous CSP lectures:

- Malte Helmert and Stefan Wölfl (summer term 2007)
- Bernhard Nebel and Stefan Wölfl (winter term 2009/10)
- Julien Hué and Stefan Wölfl (summer term 2012)

Special thanks to: Matthias Westphal, Robert Mattmüller, Gabriele Röger, Manuel Bodirsky

Literature

- Rina Dechter:
  Constraint Processing,
  Morgan Kaufmann, 2003.
- Francesca Rossi, Peter van Beek, and Toby Walsh:
  Handbook of Constraint Programming,
  Elsevier, 2006.
- Wikipedia contributors:
  Wikipedia, The Free Encyclopedia,
  `http://en.wikipedia.org/`
- Wolfram Research:
  Wolfram MathWorld,
  `http://mathworld.wolfram.com/`

- Further readings will be given during the lecture.