

Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning: Introduction

Albert-Ludwigs-Universität Freiburg



Bernhard Nebel, Stefan Wölfl, and Julien Hué

January 8, 2014

Spatio-temporal configurations can be described **quantitatively** by specifying the coordinates of the relevant objects:

Example: *At time point 10.0 object A is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object B is at position (15.2, 3.5, 23.7). Object C is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).*

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Often, however, a **qualitative** description (using a finite vocabulary) is more adequate:

Example: *Object A hit object B. Afterwards, object C arrived.*

Sometimes we want to reason with such descriptions, e.g.:

Object C was not close to object A when it hit object B.



Intention: Description of configurations using a finite vocabulary and reasoning about these descriptions

- Specification of a **vocabulary**: usually a finite set of relations (often binary) that are **pairwise disjoint** and **exhaustive**
- Specification of a **language**: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal **semantics**
- Analysis of computational properties and design of **reasoning methods** (often constraint propagation)
- Perhaps, specification of **operational semantics** for verifying whether a relation holds in a given quantitative configuration

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- ...

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

We want to talk about **time instants** (points) and binary **relations** over them.

■ **Vocabulary:**

- X equals Y : $X = Y$
- X before Y : $X < Y$
- X after Y : $X > Y$

■ **Language:**

- Allow for **disjunctions** of basic relations to express **indefinite information**. Use set of relations to express that. For instance, $\{<, =\}$ expresses \leq .
- 2^3 different relations (including the **impossible** and the **universal** relation)
- Use **sets of atomic formulae** with these relations to describe **configurations**. For example:

$$\{x\{=\}y, y\{<, >\}z\}$$

■ **Semantics:** Interpret the time point symbols and relation

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

$$\{x\{<,=\}y, y\{<,=\}z, v\{<,=\}y, w\{>\}y, z\{<,=\}x\}$$

- **Satisfiability**: Are there values for all time points such that all formulae are satisfied?
- **Satisfiability** with $v\{=\}w$?
- Finding a satisfying **instantiation** of all time points
- **Deduction**: Does $x\{=\}y$ logically follow?
Does $v\{<,=\}w$ follow?
- Finding a **minimal description**: What are the most constrained relations that describe the same set of instantiations?

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in **satisfiability**, **model finding**, and **deduction**
- **Constraint Satisfaction Problems**

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Definition

A **constraint satisfaction problem (CSP)** is given by

- a set V of n **variables** $\{v_1, \dots, v_n\}$,
- for each v_i , a **value domain** D_i
- **constraints** (relations over subsets of the variables)

Tasks:

Find one (or all) **solution(s)**, i. e., tuples

$$(d_1, \dots, d_n) \in D_1 \times \dots \times D_n$$

such that the assignment $v_i \mapsto d_i$ ($1 \leq i \leq n$) satisfies all constraints.

Motivation

CSP

Solving CSP

Qualitative
CSP

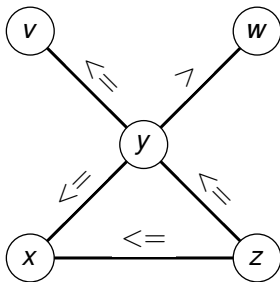
A
Pathological
Relation
System

Outlook

Literature

Our Example: Point relations

- Our point relation CSP is a binary CSP with **infinite domains**.
- It can be represented as a **constraint graph**:



Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature



k-colorability: Can we color the nodes of a graph with k colors in a way such that all nodes connected by an edge have different colors?

- The node set is the set of variables
- The domain of each variable is $\{1, \dots, k\}$
- The constraints are that nodes connected by an edge must have a different value

Note: This CSP has a particular restricted form:

- Only **binary** constraints
- The domains are **finite**

Other examples: Many problems (e.g. cross-word puzzle, n -queens problem, configuration, ...) can be cast as a CSP (and solved this way)

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Theorem

It is NP-hard to decide solvability of CSPs, even binary CSPs.

Beweis.

Since k -colorability is NP-complete (even for fixed $k \geq 3$), solvability of CSPs in general must be NP-hard. □

Question: Is CSP solvability in NP?

- **Enumeration** of all assignments and testing

↪ ... too costly

- **Backtracking** search

↪ 1001 different strategies, often “dead” search paths are explored extensively

- **Constraint propagation**: elimination of obviously impossible values followed by backtracking search

- Many other search methods, e.g., local search, stochastic search, etc.

↪ How do we solve CSP with infinite domains?

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

- Only at most **binary** constraints (i.e., we can use **constraint graph**)
- Uniform domain D for all variables
- Unary constraints D_i and binary constraints R_{ij} are **sets** of values or sets of pairs of values, resp.
- We assume that for all nodes i, j :

$$(x, y) \in R_{ij} \Rightarrow (y, x) \in R_{ji}$$

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

- A CSP is **locally consistent** if for particular subsets of the variables, solutions of the restricted CSP can be extended to solutions of a larger set of variables.
- ↪ Methods to transform a CSP into a tighter, but “equivalent” problem.

Definition

A binary CSP $\langle V, D, C \rangle$ is **arc consistent** (or **2-consistent**) if for all nodes $1 \leq i, j \leq n$,

$$x \in D_i \Rightarrow \exists y \in D_j \text{ s. t. } (x, y) \in R_{ij}$$

- ↪ When a CSP is **arc consistent**, each one variable assignment $\{v_i\} \rightarrow D$ that satisfies all (unary) constraints in v_i , i. e., D_i , can be extended to a two variable assignment $\{v_i, v_j\} \rightarrow D$ that satisfies all unary/binary constraints in these variables, i. e., D_i , D_j , and R_{ij} .

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

EnforceArcConsistency (\mathcal{C}):

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$

Output: an equivalent, but arc consistent CSP \mathcal{C}'

repeat

for each arc (v_i, v_j) with $R_{ij} \in C$

$D_i := D_i \cap \{x \in D : \text{ex. } y \in D_j \text{ s. t. } (x, y) \in R_{ij}\}$

endfor

until no domain is changed

- Terminates in time $O(n^3 \cdot k^3)$ if we have finite domains (where k is the number of values)

→ There exist different (more efficient) algorithms for enforcing arc consistency.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

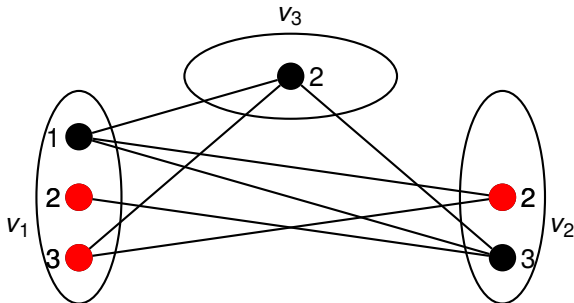
Literature

Lemma

- *Enforcing arc consistency yields an arc consistent CSP.*
- *Enforcing arc consistency is solution invariant, i. e. it does not change the set of solutions.*

↪ Arc consistent CSPs need not be consistent, and vice versa.

Arc Consistency – Example



$$D_1 = \{1, 2, 3\}$$

$$D_2 = \{2, 3\}$$

$$D_3 = \{2\}$$

$$R_{ij} = "\neq" \text{ for } i \neq j$$

■ Since all unary constraints are singletons, this is a **solution** of the CSP.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Definition

A binary CSP $\langle V, D, C \rangle$ is said to be **path consistent** (or **3-consistent**) if for all nodes $1 \leq i, j, k \leq n$,

$$x \in D_i, y \in D_j, (x, y) \in R_{ij} \Rightarrow \\ \exists z \in D_k \text{ s. t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}$$

- ↪ When a CSP is **path consistent**, each two variable assignment $\{v_i, v_j\} \rightarrow D$ satisfying all constraints in v_i and v_j can be extended to any three variable assignment $\{v_i, v_j, v_k\} \rightarrow D$ such that all constraints in these variables are satisfied.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

EnforcePathConsistency (\mathcal{C}):

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$ of size n

Output: an equivalent, but path consistent CSP \mathcal{C}'

repeat

for all $1 \leq i, j, k \leq n$

$R_{ij} := R_{ij} \cap$

$\{(x, y) : \text{ex. } z \in D_k \text{ s.t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}\}$

endfor

until no binary constraint is changed

↪ Terminates in time $O(n^5 \cdot k^5)$ if we have finite domains
(where k is the number of values)

↪ Enforcing path consistency is solution invariant.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Local Consistency (3): k -Consistency and Strong k -Consistency



Definition

- A binary CSP $\langle V, D, C \rangle$ is **k -consistent** if, given variables x_1, \dots, x_k and an assignment $a : \{x_1, \dots, x_{k-1}\} \rightarrow D$ that satisfies all constraint in these variables, a can be extended to an assignment $a' : \{x_1, \dots, x_k\} \rightarrow D$ that satisfies all constraints in these k variables.
- A binary CSP $\langle V, D, C \rangle$ is **strongly k -consistent** if it is k' -consistent for each $k' \leq k$.
- A binary CSP $\langle V, D, C \rangle$ is **globally consistent** if it is strongly n -consistent where n is the size of V .

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

- k -consistency: The computation costs grow exponentially with k .
- If a CSP is globally consistent, then
 - a solution can be constructed in polynomial time,
 - its constraints are **minimal**,
 - and it has a solution iff there is no empty constraint.
- k -consistent $\not\Rightarrow k - 1$ -consistent

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

If we want to use CSPs for qualitative reasoning, we have

- **infinite** domains
- mostly only **finitely many** relations (basic relations and their unions)
- **arc consistent** CSPs (usually)

Questions:

- How do we achieve **k -consistency** (for some fixed k)?
- Is k -consistency (for some fixed k) enough to guarantee **global consistency**?

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Composition:

$$R_1 \circ R_2 = \{(x, y) \in D^2 : \exists z \in D \text{ s.t. } (x, z) \in R_1 \text{ and } (z, y) \in R_2\}$$

Converse:

$$R^{-1} = \{(x, y) \in D^2 : (y, x) \in R\}$$

Intersection:

$$R_1 \cap R_2 = \{(x, y) \in D^2 : (x, y) \in R_1 \text{ and } (x, y) \in R_2\}$$

Union:

$$R_1 \cup R_2 = \{(x, y) \in D^2 : (x, y) \in R_1 \text{ or } (x, y) \in R_2\}$$

Complement:

$$\bar{R} = \{(x, y) \in D^2 : (x, y) \notin R\}$$

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Conditions on Vocabulary for Qualitative Reasoning

- Let **B** be a finite set of (binary) **base relations**.
- ↪ The relations in **B** should be **JEPD**, i. e., jointly exhaustive and pairwise disjoint.
- **B** should be **closed under converse**.
- Let **A** be the set of relations that can be built by taking the unions of relations from **B** ($\rightsquigarrow 2^{|B|}$ different relations).
- ↪ **A** is closed under converse, complement, intersection, union.
- **A** should be **closed under composition of base relations**, i. e., for all $B, B' \in B$, $B \circ B' \in A$.
- ↪ **A** is closed under composition of arbitrary relations.
- ↪ This condition does not hold necessarily.
Example: $B = \{<, =, >\}$ interpreted over the integers is not closed under composition (and has no finite closure):

$$< \circ < = < \setminus \{(i, j) : i = j - 1\} \subsetneq <$$

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Let **A** be a relation system over the set of **base relations** **B** that satisfies the conditions spelled out above.

↪ We may write relations as **sets** of base relations:

$$B_1 \cup \dots \cup B_n \sim \{B_1, \dots, B_n\}$$

Then the operations on the relations can be **computed** as follows:

Composition:

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m (B_i \circ B'_j)$$

Converse:

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathbf{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and **union** are defined set-theoretically.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition xRy : Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y the strongest constrained (minimal) relation entailed by the CSP.

⇒ These problems are **equivalent** under **Turing reductions**

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Beweis.

$CSAT \leq_T CENT$ and $CENT \leq_T CMIN$ are obvious.

$CENT \leq_T CSAT$: We solve $CENT$ ($CSP \models xRy?$) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let B_1, \dots, B_k be the relations for which we get a positive answer. Then $x\{B_1, \dots, B_k\}y$ is entailed by the CSP.

$CMIN \leq_T CENT$: We use entailment for computing the minimal constraint for each pair. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Given a qualitative CSP with $R_{ij} = R_{ji}^{-1}$. Then **path consistency** can be enforced by doing the following:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}).$$

Path consistency **guarantees** ...

- sometimes **minimality**
 - sometimes **satisfiability**
 - however sometimes the CSP is **not satisfiable**, even if the CSP contains only **base relations**
- All this depends on the vocabulary.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Example: Point Relations

Composition table:

	<	=	>
<	<	<	<,=,>
=	<	=	>
>	<,=,>	>	>

Abbildung: Composition table for the point algebra. For example:

$$\{<\} \circ \{=\} = \{<\}$$

- $\{<,=\} \circ \{<\} = \{<\}$
- $\{<,>\} \circ \{<\} = \{<,=,>\}$
- $\{<,=\}^{-1} = \{>,=\}$
- $\{<,=\} \cap \{>,=\} = \{=\}$

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Theorem

A path consistent CSP over the point relations is consistent.

Corollary

CSAT, CENT and CMIN are polynomial problems for the point relations.

Theorem

A path consistent CSP over all point relations without $\{<, >\}$ is minimal.

Proofs later ...

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature

Let e, d, i be (self-converse) base relations between points on a circle:

- e : Rotation by 72 degrees (left or right)
- d : Rotation by 144 degrees (left or right)
- i : Identity

Composition table:

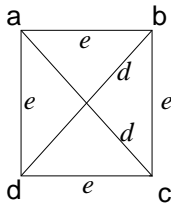
$$e \circ e = \{i, d\}$$

$$d \circ d = \{i, e\}$$

$$e \circ d = \{e, d\}$$

$$d \circ e = \{e, d\}$$

The following CSP is path consistent and contains only base relations, but it is not satisfiable:



Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature



- **Qualitative representation and reasoning** usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- **Reasoning problems** are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with **CSP methods** (but note: **infinite domains**).
- **Path consistency** is the basic reasoning step ... sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.
- Can be taken further \rightsquigarrow **relation algebra**

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature



Alan K. Mackworth.

Constraint satisfaction.

In S. C. Shapiro, editor, *Encyclopedia of Artificial Intelligence*, pages 205–211. Wiley, Chichester, England, 1987.



Alan K. Mackworth.

Consistency in networks of relations.

Artificial Intelligence, 8:99–118, 1977.



Peter B. Ladkin and Roger Maddux.

On binary constraint networks.

Journal of the ACM, 41:435–469, 1994.



Ugo Montanari.

Networks of constraints: fundamental properties and applications to picture processing.

Information Science, 7:95–132, 1974.

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature



R. Hirsch.

Tractable approximations for temporal constraint handling.
Artificial Intelligence, 116: 287-295, 2000.
(Contains a pathological set of relations.)

Motivation

CSP

Solving CSP

Qualitative
CSP

A
Pathological
Relation
System

Outlook

Literature