Principles of
Knowledge Representation and Reasoning

Nonmonotonic Reasoning III: Cumulative Logics

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel, Stefan Wölfl, and Julien Hué

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1 Introduction

- Motivation
- Properties
- Derived Rules in **C**
- Undesirable Properties

Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models).
- Nonmonotonicity is only a negative characterization:
  From $\Theta \mathrel{|\!\sim} \varphi$, it does not necessarily follow $\Theta \cup \{\psi\} \mathrel{|\!\sim} \varphi$.
- Could we have a constructive positive characterization of default reasoning?

Plausible consequences

- In classical logics, we have the logical consequence relation $\alpha \models \beta$: If α is true, then also β is true.
- Instead, we will study the relation of plausible consequence $\alpha \mathrel{|\!\sim} \beta$: If α is all we know, can we conclude β?
- $\alpha \mathrel{|\!\sim} \beta$ does not imply $\alpha \wedge \alpha' \mathrel{|\!\sim} \beta$!
  Compare to conditional probability: $P(\beta|\alpha) \neq P(\beta|\alpha, \alpha')$!
- Find rules that characterize $\mathrel{|\!\sim}$ …
  For example: if $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\sim} \gamma$, then $\alpha \mathrel{|\!\sim} \beta \wedge \gamma$.
- Write down all such rules …
- … and find a semantic characterization of $\mathrel{|\!\sim}$!

Desirable properties: Reflexivity

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Reflexivity (Ref):

$$\overline{\alpha \mathrel{\vert\!\sim} \alpha}$$

- *Rationale*: If α holds, this normally implies α.
- *Example*: Tom goes to a party normally implies that Tom goes to a party.

Reflexivity in default logic

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Let $\Delta = \langle D, W \rangle$ be a propositional default theory.
Define the relation $\mathrel{\vert\!\sim}_\Delta$ as follows:

$$\alpha \mathrel{\vert\!\sim}_\Delta \beta \iff \langle D, W \cup \{\alpha\} \rangle \mathrel{\vert\!\sim} \beta$$

$\alpha \mathrel{\vert\!\sim}_\Delta \beta$ means that β is a skeptical conclusion of $\langle D, W \cup \{\alpha\} \rangle$.

Proposition
Default logic satisfies Reflexivity.

Proof.
The question is: does α follow skeptically from $\Delta' = \langle D, W \cup \{\alpha\} \rangle$?
For each extension E of Δ', it holds $W \cup \{\alpha\} \subseteq E$ (by definition).
Hence $\alpha \in E$, and thus α belongs to all extensions of Δ'. $\qquad\square$

Desirable properties: Left Logical Equivalence

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Left Logical Equivalence (LLE):

$$\frac{\models \alpha \leftrightarrow \beta, \ \alpha \mathrel{\vert\!\sim} \gamma}{\beta \mathrel{\vert\!\sim} \gamma}$$

- *Rationale*: It is not the syntactic form, but the content that is responsible for what we conclude normally.
- *Example*: Assume that
  Tom goes or Peter goes normally implies Mary goes.
  Then we would expect that
  Peter goes or Tom goes normally implies Mary goes.

Left Logical Equivalence in default logic

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Proposition
Default logic satisfies Left Logical Equivalence.

Proof.
Assume $\models \alpha \leftrightarrow \beta$ and $\alpha \mathrel{\vert\!\sim}_\Delta \gamma$ (with $\Delta = \langle D, W \rangle$).
Hence, γ is in all extensions of $\Delta' := \langle D, W \cup \{\alpha\} \rangle$.
The definition of extensions is invariant under replacing any formula by an equivalent formula.
Thus, $\langle D, W \cup \{\beta\} \rangle$ has exactly the same extensions as Δ', and γ is in every one of them. Hence, $\beta \mathrel{\vert\!\sim}_\Delta \gamma$. $\qquad\square$

Desirable properties: Right Weakening

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Right Weakening (RW):

$$\frac{\models \alpha \to \beta,\ \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}$$

- *Rationale*: If something can be concluded normally, then everything classically implied should also be concluded normally.
- *Example*: Assume that
  `Mary goes` normally implies `Clive goes` and `John goes`.
  Then we would expect that
  `Mary goes` normally implies `Clive goes`.
- From (Ref) & (RW) Supraclassicality follows:

$$\alpha \mid\!\sim \alpha\ +\ \frac{\models \alpha \to \beta,\ \alpha \mid\!\sim \alpha}{\alpha \mid\!\sim \beta}\ \Longrightarrow\ \frac{\alpha \models \beta}{\alpha \mid\!\sim \beta}$$

Right Weakening in default logic

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Proposition

Default logic satisfies Right Weakening.

Proof.

Assume $\models \alpha \to \beta$ and $\gamma \mid\!\sim_\Delta \alpha$ (with $\Delta = \langle D, W \rangle$).
Hence, α is in each extension E of the default theory $\langle D, W \cup \{\gamma\}\rangle$.
Since extensions are closed under logical consequence, β must also be in each extension of $\langle D, W \cup \{\gamma\}\rangle$.
Hence, $\gamma \mid\!\sim_\Delta \beta$ □

Desirable properties: Cut

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Cut:

$$\frac{\alpha \mid\!\sim \beta,\ \alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma}$$

- *Rationale*: If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.
- *Example*: Assume that
  `John goes` normally implies `Mary goes`.
  Assume further that
  `John goes` and `Mary goes` normally implies `Clive goes`.
  Then we would expect that
  `John goes` normally implies `Clive goes`.

Cut in default logic

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Proposition

Default logic satisfies Cut.

Proof idea.

Assume $\alpha \mid\!\sim_\Delta \beta$ (with $\Delta = \langle D, W \rangle$). Hence β is contained in each extension of $\Delta' := \langle D, W \cup \{\alpha\}\rangle$. Show that every extension E of Δ' is also an extension of $\Delta'' = \langle D, W \cup \{\alpha \wedge \beta\}\rangle$.

- Consistency of justifications of defaults is tested against E both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \wedge \beta\}$ case.
- The preconditions that are derivable when starting from $W \cup \{\alpha\}$ are also derivable when starting from $W \cup \{\alpha \wedge \beta\}$.
- $W \cup \{\alpha \wedge \beta\}$ does not allow for deriving further preconditions because also in the $W \cup \{\alpha\}$ case at some point β is derived.

Hence, because γ belongs to all extensions of Δ'' ($\alpha \wedge \beta \mid\!\sim \gamma$), it also belongs to all extensions of Δ' ($\alpha \mid\!\sim \gamma$). □

Desirable properties: Cautious Monotonicity

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Cautious Monotonicity (CM):

$$\frac{\alpha \mathrel{\vert\!\sim} \beta,\ \alpha \mathrel{\vert\!\sim} \gamma}{\alpha \wedge \beta \mathrel{\vert\!\sim} \gamma}$$

- *Rationale*: In general, adding new premises may cancel some conclusions.
  However, existing conclusions may be added to the premises without canceling any conclusions!
- *Example*: Assume that
  `Mary goes` normally implies `Clive goes` and
  `Mary goes` normally implies `John goes`.
  `Mary goes` and `Jack goes` might not normally imply that `John goes`.
  However, `Mary goes` and `Clive goes` should normally imply that `John goes`.

Cautious Monotonicity in default logic

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Proposition

Default logic does not satisfy Cautious Monotonicity.

Proof.

Consider the default theory $\langle D, W \rangle$ with

$$D = \left\{ \frac{a:g}{g}, \frac{g:b}{b}, \frac{b:\neg g}{\neg g} \right\} \text{ and } W = \{a\}.$$

$E = \mathsf{Th}(\{a, b, g\})$ is the only extension of $\langle D, W \rangle$ and thus both b and g follow skeptically (i.e., we have $a \mathrel{\vert\!\sim}_{\langle D, \emptyset \rangle} b$ and $a \mathrel{\vert\!\sim}_{\langle D, \emptyset \rangle} g$).

For $\langle D, \{a \wedge b\} \rangle$ also $\mathsf{Th}(\{a, b, \neg g\})$ is an extension, and thus g does not follow skeptically (i.e., $a \wedge b \mathrel{\not\vert\!\sim}_{\langle D, \emptyset \rangle} g$). $\qquad\square$

Cumulativity

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Lemma

Rules (Cut) & (CM) can be equivalently stated as follows:

If $\alpha \mathrel{\vert\!\sim} \beta$, then the sets of plausible conclusions from α and $\alpha \wedge \beta$ are identical.

This property is called Cumulativity.

Proof.

$\Rightarrow$: Assume that we may apply both rules (Cut) and (CM) and assume $\alpha \mathrel{\vert\!\sim} \beta$.
Assume further that $\alpha \mathrel{\vert\!\sim} \gamma$. By applying (CM), we obtain $\alpha \wedge \beta \mathrel{\vert\!\sim} \gamma$.
Similarly, by applying (Cut), from $\alpha \wedge \beta \mathrel{\vert\!\sim} \gamma$ it follows $\alpha \mathrel{\vert\!\sim} \gamma$.
Hence the plausible conclusions from α and $\alpha \wedge \beta$ are the same.
$\Leftarrow$: Assume Cumulativity and $\alpha \mathrel{\vert\!\sim} \beta$. Now we can derive both rules (Cut) and (CM). $\qquad\square$

The System **C**

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1. Reflexivity

$$\overline{\alpha \mathrel{\vert\!\sim} \alpha}$$

2. Left Logical Equivalence

$$\frac{\models \alpha \leftrightarrow \beta,\ \alpha \mathrel{\vert\!\sim} \gamma}{\beta \mathrel{\vert\!\sim} \gamma}$$

3. Right Weakening

$$\frac{\models \alpha \rightarrow \beta,\ \gamma \mathrel{\vert\!\sim} \alpha}{\gamma \mathrel{\vert\!\sim} \beta}$$

4. Cut

$$\frac{\alpha \mathrel{\vert\!\sim} \beta,\ \alpha \wedge \beta \mathrel{\vert\!\sim} \gamma}{\alpha \mathrel{\vert\!\sim} \gamma}$$

5. Cautious Monotonicity

$$\frac{\alpha \mathrel{\vert\!\sim} \beta,\ \alpha \mathrel{\vert\!\sim} \gamma}{\alpha \wedge \beta \mathrel{\vert\!\sim} \gamma}$$

Derived rules in **C**

- Equivalence:

$$\frac{\alpha \vdash\!\!\sim \beta, \ \beta \vdash\!\!\sim \alpha, \ \alpha \vdash\!\!\sim \gamma}{\beta \vdash\!\!\sim \gamma}$$

- And:

$$\frac{\alpha \vdash\!\!\sim \beta, \ \alpha \vdash\!\!\sim \gamma}{\alpha \vdash\!\!\sim \beta \wedge \gamma}$$

- MPC:

$$\frac{\alpha \vdash\!\!\sim \beta \rightarrow \gamma, \ \alpha \vdash\!\!\sim \beta}{\alpha \vdash\!\!\sim \gamma}$$

Derived rules: proofs

Proof (Equivalence).

Assumption:	$\alpha \vdash\!\!\sim \beta, \ \beta \vdash\!\!\sim \alpha, \ \alpha \vdash\!\!\sim \gamma$
Cautious Monotonicity:	$\alpha \wedge \beta \vdash\!\!\sim \gamma$
Left L Equivalence:	$\beta \wedge \alpha \vdash\!\!\sim \gamma$
Cut:	$\beta \vdash\!\!\sim \gamma$

□

Proof (And).

Assumption:	$\alpha \vdash\!\!\sim \beta, \ \alpha \vdash\!\!\sim \gamma$
Cautious Monotonicity:	$\alpha \wedge \beta \vdash\!\!\sim \gamma$
propositional logic:	$\alpha \wedge \beta \wedge \gamma \models \beta \wedge \gamma$
Supraclassicality:	$\alpha \wedge \beta \wedge \gamma \vdash\!\!\sim \beta \wedge \gamma$
Cut:	$\alpha \wedge \beta \vdash\!\!\sim \beta \wedge \gamma$
Cut:	$\alpha \vdash\!\!\sim \beta \wedge \gamma$

□

MPC is an exercise.

Undesirable properties: Monotonicity and Contraposition

- Monotonicity:

$$\frac{\models \alpha \rightarrow \beta, \ \beta \vdash\!\!\sim \gamma}{\alpha \vdash\!\!\sim \gamma}$$

  - *Example*: Let us assume that
    `John goes` normally implies `Mary goes`.
    Now we will probably not expect that
    `John goes` and `Joan` (who is not in talking terms with
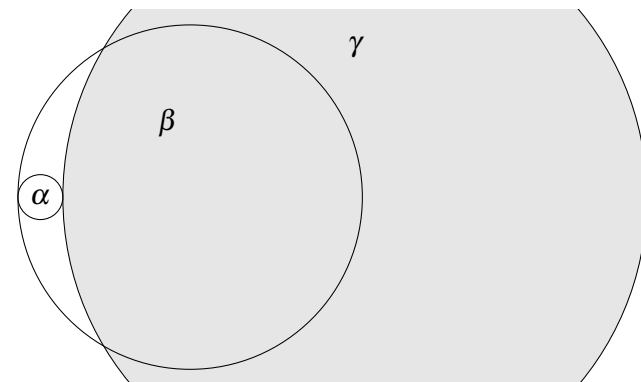   `Mary`) `goes` normally implies `Mary goes`.

- Contraposition:

$$\frac{\alpha \vdash\!\!\sim \beta}{\neg \beta \vdash\!\!\sim \neg \alpha}$$

  - *Example*: Let us assume that
    `John goes` normally implies `Mary goes`.
    Would we expect that
    `Mary does not go` normally implies `John does not go`?
    What if `John goes` always?

Undesirable properties: Monotonicity

$\alpha \models \beta, \ \beta \vdash\!\!\sim \gamma$, but not $\alpha \vdash\!\!\sim \gamma$ — pictorially:

Undesirable properties: Contraposition

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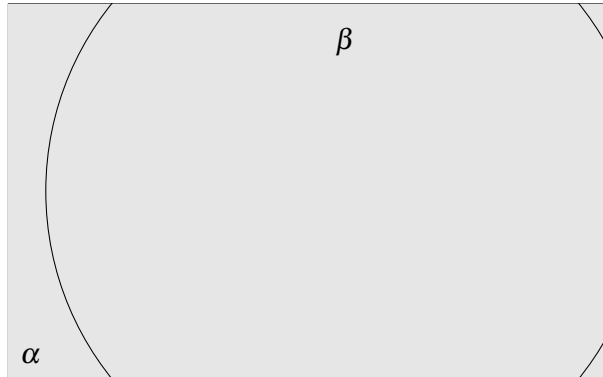Reasoning

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$\alpha \mathrel{\vdash\!\sim} \beta$, but not $\neg\beta \mathrel{\vdash\!\sim} \neg\alpha$ — pictorially:

β

α

Undesirable properties: Transitivity & EHD

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- Transitivity:
$$\frac{\alpha \mathrel{\vdash\!\sim} \beta,\ \beta \mathrel{\vdash\!\sim} \gamma}{\alpha \mathrel{\vdash\!\sim} \gamma}$$

  - *Example*: Let us assume that
    John goes normally implies Mary goes and
    Mary goes normally implies Jack goes.
    Now, should John goes normally imply that Jack goes?
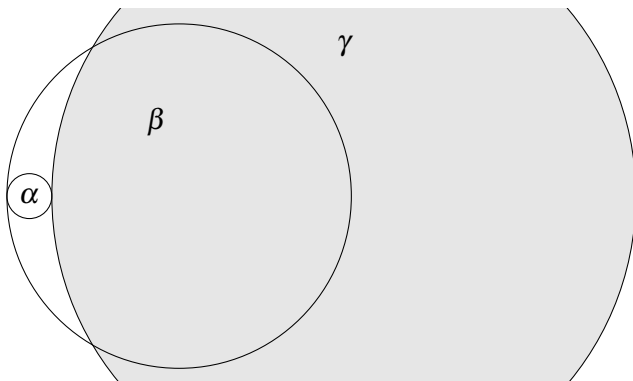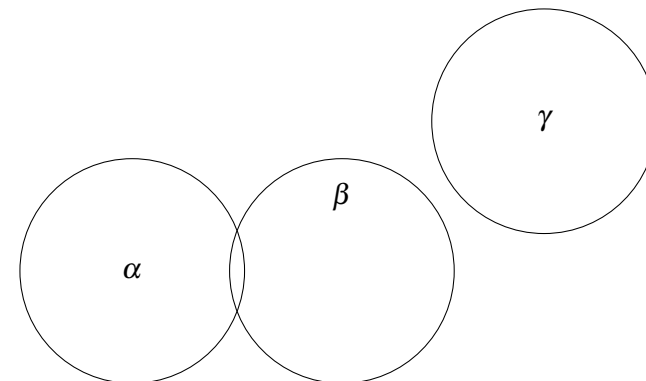   What, if John goes very seldom?

- Easy Half of the Deduction Theorem (EHD):

$$\frac{\alpha \mathrel{\vdash\!\sim} \beta \to \gamma}{\alpha \wedge \beta \mathrel{\vdash\!\sim} \gamma}$$

Undesirable properties: Transitivity

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$\alpha \mathrel{\vdash\!\sim} \beta,\ \beta \mathrel{\vdash\!\sim} \gamma$, but not $\alpha \mathrel{\vdash\!\sim} \gamma$ — pictorially:

γ

β

α

Undesirable properties: EHD

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$\alpha \mathrel{\vdash\!\sim} \beta \to \gamma$, but not $\alpha \wedge \beta \mathrel{\vdash\!\sim} \gamma$ — pictorially:

γ

β

α

Monotonicity vs EHD

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Theorem
*In the presence of the rules in system **C**, the rules Monotonicity and EHD are equivalent.*

Proof.
Monotonicity $\Rightarrow$ EHD:

- $\alpha \mid\!\sim \beta \to \gamma$ (assumption)
- $\alpha \wedge \beta \mid\!\sim \beta \to \gamma$ (Monotonicity)
- $\alpha \wedge \beta \mid\!\sim \alpha \wedge \beta$ (Ref)
- $\alpha \wedge \beta \mid\!\sim \beta$ (RW)
- $\alpha \wedge \beta \mid\!\sim \gamma$ (MPC)

Monotonicity $\Leftarrow$ EHD:

- $\models \alpha \to \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\beta \mid\!\sim \alpha \to \gamma$ (RW)
- $\beta \wedge \alpha \mid\!\sim \gamma$ (EHD)
- $\alpha \mid\!\sim \gamma$ (LLE)

$\square$

Monotonicity vs Transitivity

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Theorem
*In the presence of the rules in system **C**, the rules Monotonicity and Transitivity are equivalent.*

Proof.
Monotonicity $\Rightarrow$ Transitivity:

- $\alpha \mid\!\sim \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\alpha \wedge \beta \mid\!\sim \gamma$ (Monotonicity)
- $\alpha \mid\!\sim \gamma$ (Cut)

Monotonicity $\Leftarrow$ Transitivity:

- $\models \alpha \to \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \mid\!\sim \beta$ (Supraclassicality)
- $\alpha \mid\!\sim \gamma$ (Transitivity)

$\square$

Contraposition?

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Theorem
In the presence of Right Weakening, Contraposition implies Monotonicity.

Proof.
- $\models \alpha \to \beta, \beta \mid\!\sim \gamma$ (assumption)
- $\neg\gamma \mid\!\sim \neg\beta$ (Contraposition)
- $\models \neg\beta \to \neg\alpha$ (classical contraposition)
- $\neg\gamma \mid\!\sim \neg\alpha$ (RW)
- $\alpha \mid\!\sim \gamma$ (Contraposition)

$\square$

Note: Monotonicity does not imply Contraposition, even in the presence of all rules of system **C**!

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2 Reasoning

Reasoning with conditionals

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- How do we reason with $\mathrel|\!\sim$ from φ to ψ?
- *Assumption*: We have some (finite) set K of conditional statements of the form $\alpha \mathrel|\!\sim \beta$.
  The question is: Assuming the statements in K, is it plausible to conclude ψ given φ?
- *Idea*: We consider all cumulative consequence relations that contain K.
  Cumulative consequence relation: any relation $\mathrel|\!\sim$ between propositional logic formulae that is closed under the rules of system **C**.
- Remark: It suffices to consider only the minimal cumulative consequence relation containing $K \ldots$

Cumulative closure

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Lemma

The set of cumulative consequence relations is closed under (arbitrary) intersections.

Proof.

Let $\mathrel|\!\sim_1$ and $\mathrel|\!\sim_2$ be cumulative consequence relations. We have to show that $\mathrel|\!\sim_1 \cap \mathrel|\!\sim_2$ is a cumulative consequence relation, that is, it is closed under all the rules of system **C**.
Take any instance of any of the rules. If the preconditions are satisfied by $\mathrel|\!\sim_1$ and $\mathrel|\!\sim_2$, then the consequence is trivially also satisfied by both. A similar argument works if we consider an arbitrary family of consequence relations. □

Cumulative closure

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Theorem

For each set of conditional statements K, there exists a unique minimal cumulative consequence relation containing K.

Proof.

From the previous lemma it is clear that the intersection of all the cumulative consequence relations containing K is already such a cumulative consequence relation.
Obviously, there cannot be two distinct such minimal relations. □

This relation is called the cumulative closure K^C of K.

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3 Semantics

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- Cumulative Models
- Consequence Relations

Cumulative models – informally

- We will now try to characterize cumulative reasoning model-theoretically.

- *Idea*: Cumulative models consist of states ordered by a preference relation.

- States characterize beliefs.

- The preference relation, $\prec$, expresses the normality of the beliefs.
  We read $s \prec t$ as: state s is preferred to/more normal than state t.

- We say: $\alpha \mathrel{|\!\sim} \beta$ is accepted in a model if in all most preferred states in which α is true also β is true.

Preference relation

We consider an arbitrary binary relation $\prec$ on a given set of states S.

Later, we will assume that $\prec$ has particular properties, e.g., that $\prec$ is irreflexive, asymmetric, transitive, a partial order, . . .
. . . but currently we make no such restrictions.

We need a condition on state sets claiming that each state is, or is related to, a most preferred state.

Definition (Smoothness)

Let $P \subseteq S$.

- We say that $s \in P$ is minimal in P if $s' \not\prec s$ for each $s' \in P$.
- P is called smooth if for each $s \in P$, either s is minimal in P or there exists an s' such that s' is minimal in P and $s' \prec s$.

Cumulative Models – formally

Let $\mathcal{U}$ be the set of all possible worlds (i.e., propositional interpretations).

- A cumulative model is a triple $W = \langle S, I, \prec \rangle$ such that
  1. S is a set of states,
  2. I is a mapping $I : S \to 2^{\mathcal{U}}$, and
  3. $\prec$ is an arbitrary binary relation on S

  such that the smoothness condition is satisfied (see below).

- A state $s \in S$ satisfies a formula α ($s \models \alpha$) if $m \models \alpha$ for each propositional interpretation $m \in I(s)$.
  The set of states satisfying α is denoted by $\widehat{\alpha}$.

- Smoothness condition: A cumulative model satisfies this condition if for all formulae α, $\widehat{\alpha}$ is smooth.

Consequence relation induced by a cumulative model

A cumulative model W induces a consequence relation $\mathrel{|\!\sim}_W$ as follows:

$$\alpha \mathrel{|\!\sim}_W \beta \text{ iff } s \models \beta \text{ for every minimal } s \text{ in } \widehat{\alpha}.$$

Example

Model $W = \langle \{s_1, s_2, s_3\}, I, \prec \rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

$$I(s_1) = \{\{\neg p, b, f\}\}$$
$$I(s_2) = \{\{p, b, \neg f\}\}$$
$$I(s_3) = \{\{\neg p, \neg b, f\}, \{\neg p, \neg b, \neg f\}\}$$

Does W satisfy the smoothness condition?
$\neg p \wedge \neg b \mathrel{|\!\sim} f$?   N   Also: $\neg p \wedge \neg b \mathrel{|\!\not\sim} \neg f$!
$p \mathrel{|\!\sim} \neg f$?   Y
$\neg p \mathrel{|\!\sim} f$?   Y

Soundness 1

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Theorem

If W is a cumulative model, then $\mathrel{\vdash}_W$ is a cumulative consequence relation.

Proof.

- Reflexivity: satisfied.

- LLE: satisfied.

- RW: satisfied.

- Cut: $\alpha \mathrel{\vdash}_W \beta$, $\alpha \wedge \beta \mathrel{\vdash}_W \gamma \Rightarrow \alpha \mathrel{\vdash}_W \gamma$. Assume that all minimal elements of $\widehat{\alpha}$ satisfy β, and all minimal elements of $\widehat{\alpha \wedge \beta}$ satisfy γ. Every minimal element of $\widehat{\alpha}$ satisfies $\alpha \wedge \beta$. Since $\widehat{\alpha \wedge \beta} \subseteq \widehat{\alpha}$, all minimal elements of $\widehat{\alpha}$ are also minimal elements of $\widehat{\alpha \wedge \beta}$. Hence $\alpha \mathrel{\vdash}_W \gamma$.

Soundness 2

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Proof continues...

- Cautious Monotonicity: $(\alpha \mathrel{\vdash} \beta, \alpha \mathrel{\vdash} \gamma \Rightarrow \alpha \wedge \beta \mathrel{\vdash} \gamma)$

  Assume $\alpha \mathrel{\vdash}_W \beta$ and $\alpha \mathrel{\vdash}_W \gamma$. We have to show: $\alpha \wedge \beta \mathrel{\vdash}_W \gamma$, i.e., $s \models \gamma$ for all minimal $s \in \widehat{\alpha \wedge \beta}$.

  Clearly, every minimal $s \in \widehat{\alpha \wedge \beta}$ is in $\widehat{\alpha}$.

  We show that every minimal $s \in \widehat{\alpha \wedge \beta}$ is minimal in $\widehat{\alpha}$.

  Assumption: There is s that is minimal in $\widehat{\alpha \wedge \beta}$, but not minimal in $\widehat{\alpha}$. Because of smoothness there is minimal $s' \in \widehat{\alpha}$ such that $s' \prec s$. We know, however, that $s' \models \beta$, which means that $s' \in \widehat{\alpha \wedge \beta}$. Hence s is not minimal in $\widehat{\alpha \wedge \beta}$. Contradiction!

  Hence s must be minimal in $\widehat{\alpha}$, and therefore $s \models \gamma$. Because this is true for all minimal elements in $\widehat{\alpha \wedge \beta}$, we get $\alpha \wedge \beta \mathrel{\vdash}_W \gamma$. $\quad\square$

Consequence: counterexamples

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Now we have a method for showing that a principle does not hold for cumulative consequence relations:

... construct a cumulative model that falsifies the principle.

Contraposition: $\alpha \mathrel{\vdash} \beta \Rightarrow \neg\beta \mathrel{\vdash} \neg\alpha$

$$W = \langle S, I, \prec \rangle$$
$$S = \{s_1, s_2\}$$
$$s_i \not\prec s_j \; \forall s_i, s_j \in S$$
$$I(s_1) = \big\{\{a, b\}\big\}$$
$$I(s_2) = \big\{\{a, \neg b\}, \{\neg a, \neg b\}\big\}$$

W is a cumulative model with $a \mathrel{\vdash}_W b$, but $\neg b \mathrel{\not\vdash}_W \neg a$.

Completeness?

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- Each cumulative model *W* induces a cumulative consequence relation $\mathrel{\vdash}_W$.

- Problem: Can we generate all cumulative consequence relations in this way?

- We can! There is a representation theorem:

Theorem (Representation of cumulative consequence)

A consequence relation is cumulative if and only if it is induced by some cumulative model.

⤳ Cumulative consequence can be characterized independently from the set of inference rules.

Transitivity of the preference relation?

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- Could we strengthen the preference relation to transitive relations without sacrificing anything?
  No!

- In such models, the following additional principle called Loop is valid:

$$\frac{\alpha_0 \mathrel{|\!\sim} \alpha_1, \alpha_1 \mathrel{|\!\sim} \alpha_2, \ldots, \alpha_k \mathrel{|\!\sim} \alpha_0}{\alpha_0 \mathrel{|\!\sim} \alpha_k}$$

- For the system $\mathbf{CL} = \mathbf{C} + \text{(Loop)}$ and cumulative models with transitive preference relations, we could prove another representation theorem.

The Or Rule

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Or rule:

$$\frac{\alpha \mathrel{|\!\sim} \gamma, \; \beta \mathrel{|\!\sim} \gamma}{\alpha \vee \beta \mathrel{|\!\sim} \gamma}$$

Not valid in system **C**. Counterexample:

$$
\begin{aligned}
W &= \langle S, I, \prec \rangle \\
S &= \{s_1, s_2, s_3\}, s_i \not\prec s_j \; \forall s_i, s_j \in S \\
I(s_1) &= \big\{\{a,b,c\},\{a,\neg b,c\}\big\} \\
I(s_2) &= \big\{\{a,b,c\},\{\neg a,b,c\}\big\} \\
I(s_3) &= \big\{\{a,b,\neg c\},\{a,\neg b,\neg c\},\{\neg a,b,\neg c\}\big\}
\end{aligned}
$$

$a \mathrel{|\!\sim}_W c$, $b \mathrel{|\!\sim}_W c$, but not $a \vee b \mathrel{|\!\sim}_W c$.

Note: Or is not valid in default logic.

---

4 Preferential Reasoning

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- Preferential Relations

System **P**

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- System **P** contains all rules of **C** and the Or rule.
- A consequence relation that satisfies **P** is called preferential.
- Derived rules in **P**:
  - Hard half of the deduction theorem (S):

$$\frac{\alpha \wedge \beta \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \beta \to \gamma}$$

  - Proof by case analysis (D):

$$\frac{\alpha \wedge \neg \beta \mathrel{|\!\sim} \gamma, \; \alpha \wedge \beta \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \gamma}$$

- D and Or are equivalent in the presence of the rules in C.

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Definition

A cumulative model $W = \langle S, I, \prec \rangle$ such that $\prec$ is a strict partial order (irreflexive and transitive) and $|I(s)| = 1$ for all $s \in S$ is called a preferential model.

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Theorem (Soundness)

The consequence relation $\vdash\!\sim_W$ induced by a preferential model is preferential.

Proof.

Since W is cumulative, we only have to verify that Or holds. Note that in preferential models we have $\widehat{\alpha \vee \beta} = \widehat{\alpha} \cup \widehat{\beta}$. Suppose $\alpha \vdash\!\sim_W \gamma$ and $\beta \vdash\!\sim_W \gamma$. Because of the above equation, each minimal state of $\widehat{\alpha \vee \beta}$ is minimal in $\widehat{\alpha} \cup \widehat{\beta}$. Since γ is satisfied in all minimal states in $\widehat{\alpha} \cup \widehat{\beta}$, γ is also satisfied in all minimal states of $\widehat{\alpha \vee \beta}$. Hence $\alpha \vee \beta \vdash\!\sim_W \gamma$. □

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Theorem (Representation of preferential consequence)

A consequence relation is preferential if and only if it is induced by a preferential model.

Proof.

Similar to the one for **C**. □

Summary of cumulative systems

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System	Models
C	
Reflexivity	States: sets of worlds
Left Logical Equivalence	Preference relation: arbitrary
Right Weakening	Models must be smooth
Cut	
Cautious Monotonicity	
CL	
+ Loop	Preference relation: strict partial order
P	
+ Or	States: singletons

Strengthening the consequence relation

- System **C** and System **P** do not produce many of the inferences one would hope for:

    Given K = {Bird $\mid\sim$ Flies} one cannot conclude Red $\wedge$ Bird $\mid\sim$ Flies!

- In general, adding information that is irrelevant cancels the plausible conclusions.
    $\implies$ Cumulative and Preferential consequence relations are too nonmonotonic.

- The plausible conclusions have to be strengthened!

Strengthening the consequence relations

- The rules so far seem to be reasonable: one cannot think of rules of the same form (if we have some plausible implications, other plausible implications should hold) that could be added.
- However, there are other types of rules one might want add.
- Disjunctive Rationality:

$$\frac{\alpha \not\mid\sim \gamma, \; \beta \not\mid\sim \gamma}{\alpha \vee \beta \not\mid\sim \gamma}$$

- Rational Monotonicity:

$$\frac{\alpha \mid\sim \gamma, \; \alpha \not\mid\sim \neg\beta}{\alpha \wedge \beta \mid\sim \gamma}$$

- Note: Consequence relations obeying these rules are not closed under intersection, which is a problem.

Summary

- Instead of ad hoc extensions of the logical machinery, analyze the properties of nonmonotonic consequence relations.
- Correspondence between rule system and models for System **C**, and for System **P** could also be established wrt. a probabilistic semantics.
- Irrelevant information poses a problem. Solution approaches: rational monotonicity, maximum entropy approach.

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5 Literature

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Literature

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