

Principles of AI Planning

10. Planning as search: abstractions

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Coming up with heuristics in a principled way

General procedure for obtaining a heuristic

Solve an easier version of the problem.

Two common methods:

- **relaxation**: consider **less constrained** version of the problem
- **abstraction**: consider **smaller** version of real problem

In previous chapters, we have studied **relaxation**, which has been very successfully applied to **satisficing planning**.

Now, we study **abstraction**, which is one of the most prominent techniques for **optimal planning**.

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Abstracting a transition system

Abstracting a transition system means **dropping some distinctions** between states, while **preserving the transition behaviour** as much as possible.

- An abstraction of a transition system \mathcal{T} is defined by an **abstraction mapping** α that defines which states of \mathcal{T} should be distinguished and which ones should not.
- From \mathcal{T} and α , we compute an **abstract transition system** \mathcal{T}' which is similar to \mathcal{T} , but smaller.
- The **abstract goal distances** (goal distances in \mathcal{T}') are used as heuristic estimates for goal distances in \mathcal{T} .

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Abstracting a transition system: example

Example (15-puzzle)

A **15-puzzle** state is given by a permutation $\langle b, t_1, \dots, t_{15} \rangle$ of $\{1, \dots, 16\}$, where b denotes the blank position and the other components denote the positions of the 15 tiles.

One possible **abstraction mapping** ignores the precise location of tiles 8–15, i. e., two states are distinguished iff they differ in the position of the blank or one of the tiles 1–7:

$$\alpha(\langle b, t_1, \dots, t_{15} \rangle) = \langle b, t_1, \dots, t_7 \rangle$$

The heuristic values for this abstraction correspond to the cost of moving tiles 1–7 to their goal positions.

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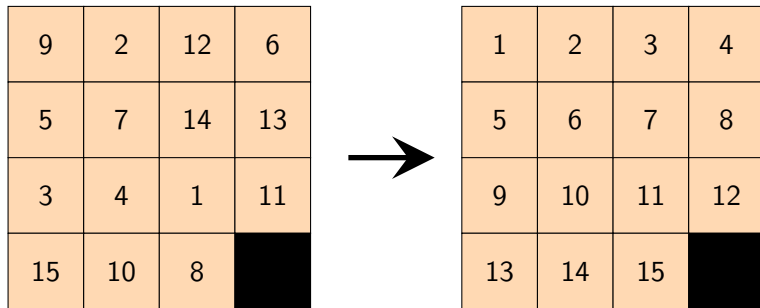
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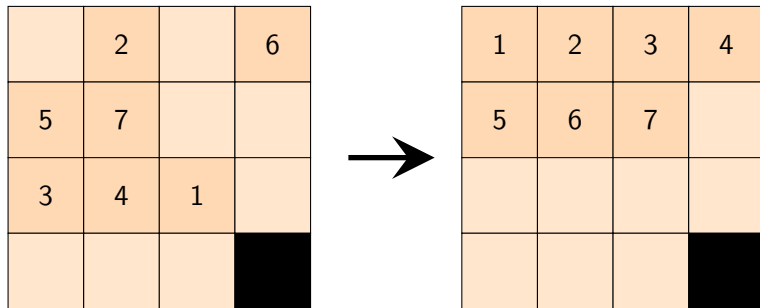
Abstraction example: 15-puzzle



real state space

- $16! = 20922789888000 \approx 2 \cdot 10^{13}$ states
- $\frac{16!}{2} = 10461394944000 \approx 10^{13}$ reachable states

Abstraction example: 15-puzzle



abstract state space

- $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$ states
- $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$ reachable states

Computing the abstract transition system

Given \mathcal{T} and α , how do we compute \mathcal{T}' ?

Requirement

We want to obtain an **admissible heuristic**.

Hence, $h^*(\alpha(s))$ (in the abstract state space \mathcal{T}') should never overestimate $h^*(s)$ (in the concrete state space \mathcal{T}).

An easy way to achieve this is to ensure that **all solutions in \mathcal{T} also exist in \mathcal{T}'** :

- If s is a goal state in \mathcal{T} , then $\alpha(s)$ is a goal state in \mathcal{T}' .
- If \mathcal{T} has a transition from s to t , then \mathcal{T}' has a transition from $\alpha(s)$ to $\alpha(t)$.

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Computing the abstract transition system: example

Example (15-puzzle)

In the running example:

- \mathcal{T} has the unique goal state $\langle 16, 1, 2, \dots, 15 \rangle$.
 - ↪ \mathcal{T}' has the unique goal state $\langle 16, 1, 2, \dots, 7 \rangle$.
- Let x and y be neighboring positions in the 4×4 grid.
 \mathcal{T} has a transition from $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_{15} \rangle$ to $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_{15} \rangle$ for all $i \in \{1, \dots, 15\}$.
 - ↪ \mathcal{T}' has a transition from $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_7 \rangle$ to $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_7 \rangle$ for all $i \in \{1, \dots, 7\}$.
 - ↪ Moreover, \mathcal{T}' has a transition from $\langle x, t_1, \dots, t_7 \rangle$ to $\langle y, t_1, \dots, t_7 \rangle$ if $y \notin \{t_1, \dots, t_7\}$.

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Practical requirements for abstractions

To be useful in practice, an abstraction heuristic must be efficiently computable. This gives us two requirements for α :

- For a given state s , the **abstract state** $\alpha(s)$ must be efficiently computable.
- For a given abstract state $\alpha(s)$, the **abstract goal distance** $h^*(\alpha(s))$ must be efficiently computable.

There are different ways of achieving these requirements:

- **pattern database heuristics** (Culberson & Schaeffer, 1996)
- **merge-and-shrink abstractions** (Dräger, Finkbeiner & Podelski, 2006)
- **structural patterns** (Katz & Domshlak, 2008b)
 - not covered in this course

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Practical requirements for abstractions: example

Example (15-puzzle)

In our running example, α can be very efficiently computed: just project the given 16-tuple to its first 8 components.

To compute abstract goal distances efficiently during search, most common algorithms precompute **all abstract goal distances** prior to search by performing a backward breadth-first search from the goal state(s). The distances are then stored in a table (requires about 495 MB of RAM).

During search, computing $h^*(\alpha(s))$ is just a table lookup.

This heuristic is an example of a **pattern database heuristic**.

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Multiple abstractions

- One important practical question is how to come up with a suitable abstraction mapping α .
- Indeed, there is usually a **huge number of possibilities**, and it is important to pick good abstractions (i. e., ones that lead to informative heuristics).
- However, it is generally **not necessary to commit to a single abstraction**.

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Combining multiple abstractions

Maximizing several abstractions:

- Each abstraction mapping gives rise to an admissible heuristic.
- By computing the **maximum** of several admissible heuristics, we obtain another admissible heuristic which **dominates** the component heuristics.
- Thus, we can always compute several abstractions and maximize over the individual abstract goal distances.

Adding several abstractions:

- In some cases, we can even compute the **sum** of individual estimates and still stay admissible.
- Summation often leads to **much higher estimates** than maximization, so it is **important to understand when it is admissible**.

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Maximizing several abstractions: example

Example (15-puzzle)

- mapping to tiles 1–7 was arbitrary
~> can use **any subset** of tiles
- with the same amount of memory required for the tables for the mapping to tiles 1–7, we could store the tables for **nine different abstractions** to six tiles and the blank
- use **maximum** of individual estimates

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Adding several abstractions: example

9	2	12	6
5	7	14	13
3	4	1	11
15	10	8	

9	2	12	6
5	7	14	13
3	4	1	11
15	10	8	

- **1st abstraction:** ignore precise location of 8–15
 - **2nd abstraction:** ignore precise location of 1–7
- ↪ Is the **sum** of the abstraction heuristics **admissible**?

Adding several abstractions: example

	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

- **1st abstraction:** ignore precise location of 8–15
 - **2nd abstraction:** ignore precise location of 1–7
- ↪ The **sum** of the abstraction heuristics is **not admissible**.

Adding several abstractions: example

	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

- **1st abstraction:** ignore precise location of 8–15 and blank
 - **2nd abstraction:** ignore precise location of 1–7 and blank
- ↪ The **sum** of the abstraction heuristics is **admissible**.

Our plan for the next lectures

In the following, we take a deeper look at abstractions and their use for admissible heuristics.

- In the rest of **this chapter**, we **formally introduce** abstractions and abstraction heuristics and study some of their most important properties.
- In the **following chapters**, we discuss some particular classes of abstraction heuristics in detail, namely **pattern database heuristics** and **merge-and-shrink abstractions**.

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Transition systems

Reminder from Chapter 2:

Definition (transition system)

A **transition system** is a 5-tuple $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$ where

- S is a finite set of **states**,
- L is a finite set of (transition) **labels**,
- $T \subseteq S \times L \times S$ is the **transition relation**,
- $s_0 \in S$ is the **initial state**, and
- $S_\star \subseteq S$ is the set of **goal states**.

We say that \mathcal{T} **has the transition** $\langle s, \ell, s' \rangle$ if $\langle s, \ell, s' \rangle \in T$.

We also write this $s \xrightarrow{\ell} s'$, or $s \rightarrow s'$ when not interested in ℓ .

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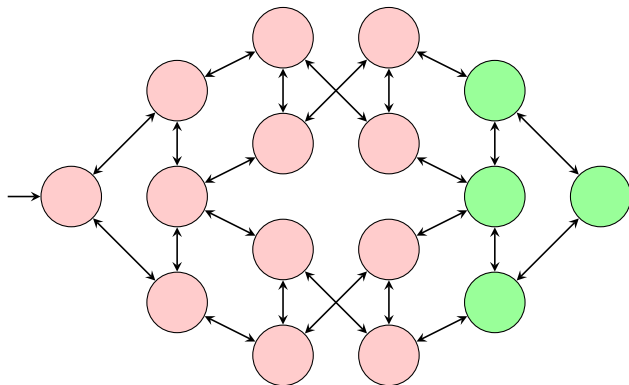
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Transition systems: example



Note: To reduce clutter, our figures usually omit arc labels and collapse transitions between identical states. However, these are important for the formal definition of the transition system.

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Transition systems of FDR planning tasks

Definition (induced transition system of an FDR planning task)

Let $\Pi = \langle V, I, O, \gamma \rangle$ be an FDR planning task.

The **induced transition system of Π** , in symbols $\mathcal{T}(\Pi)$, is the transition system $\mathcal{T}(\Pi) = \langle S, L, T, s_0, S_\star \rangle$, where

- S is the set of states over V ,
- $L = O$,
- $T = \{ \langle s, o, t \rangle \in S \times L \times S \mid \text{app}_o(s) = t \}$,
- $s_0 = I$, and
- $S_\star = \{ s \in S \mid s \models \gamma \}$.

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Example task: one package, two trucks

Example (one package, two trucks)

Consider the following FDR planning task $\langle V, I, O, \gamma \rangle$:

- $V = \{p, t_A, t_B\}$ with
 - $\mathcal{D}_p = \{L, R, A, B\}$
 - $\mathcal{D}_{t_A} = \mathcal{D}_{t_B} = \{L, R\}$
- $I = \{p \mapsto L, t_A \mapsto R, t_B \mapsto R\}$
- $O = \{\text{pickup}_{i,j} \mid i \in \{A, B\}, j \in \{L, R\}\}$
 $\cup \{\text{drop}_{i,j} \mid i \in \{A, B\}, j \in \{L, R\}\}$
 $\cup \{\text{move}_{i,j,j'} \mid i \in \{A, B\}, j, j' \in \{L, R\}, j \neq j'\}$, where
 - $\text{pickup}_{i,j} = \langle t_i = j \wedge p = j, p := i \rangle$
 - $\text{drop}_{i,j} = \langle t_i = j \wedge p = i, p := j \rangle$
 - $\text{move}_{i,j,j'} = \langle t_i = j, t_i := j' \rangle$
- $\gamma = (p = R)$

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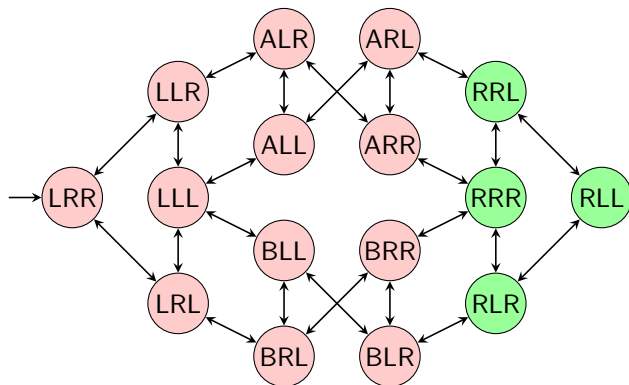
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Transition system of example task



- State $\{p \mapsto i, t_A \mapsto j, t_B \mapsto k\}$ is depicted as ijk .
- Transition labels are again not shown. For example, the transition from LLL to ALL has the label $\text{pickup}_{A,L}$.

Definition (abstraction, abstraction mapping)

Let $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$ and $\mathcal{T}' = \langle S', L', T', s'_0, S'_\star \rangle$ be transition systems with the same label set $L = L'$, and let $\alpha : S \rightarrow S'$ be a **surjective** function.

We say that \mathcal{T}' is **an abstraction of \mathcal{T} with abstraction mapping α** (or: **abstraction function α**) if

- $\alpha(s_0) = s'_0$,
- for all $s \in S_\star$, we have $\alpha(s) \in S'_\star$, and
- for all $\langle s, \ell, t \rangle \in T$, we have $\langle \alpha(s), \ell, \alpha(t) \rangle \in T'$.

Abstractions: terminology

Let \mathcal{T} and \mathcal{T}' be transition systems and α a function such that \mathcal{T}' is an abstraction of \mathcal{T} with abstraction mapping α .

- \mathcal{T} is called the **concrete transition system**.
- \mathcal{T}' is called the **abstract transition system**.
- Similarly: **concrete/abstract state space**, **concrete/abstract transition**, etc.

We say that:

- \mathcal{T}' is an **abstraction of \mathcal{T}** (without mentioning α)
- α is an **abstraction mapping on \mathcal{T}** (without mentioning \mathcal{T}')

Note: For a given \mathcal{T} and α , there can be multiple abstractions \mathcal{T}' , and for a given \mathcal{T} and \mathcal{T}' , there can be multiple abstraction mappings α .

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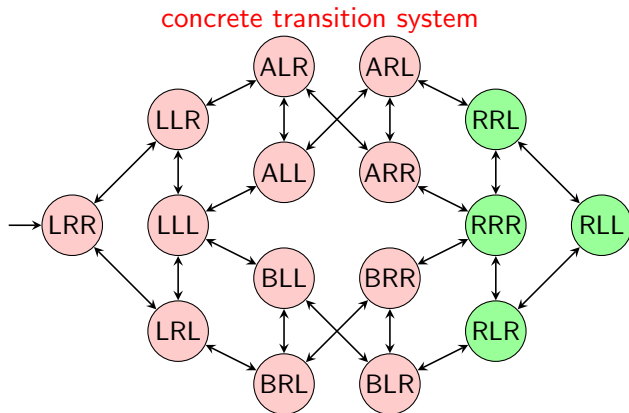
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Abstraction: example



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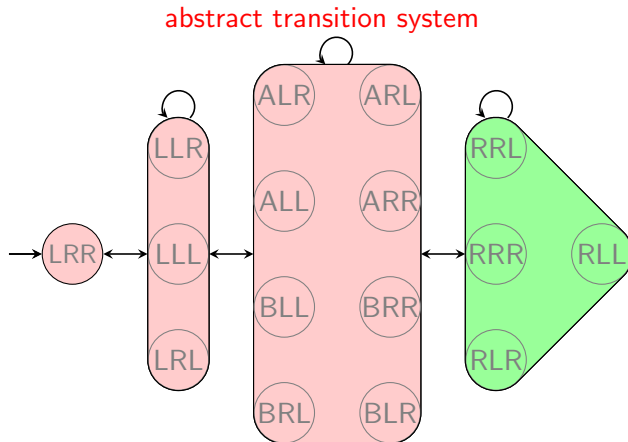
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Abstraction: example



Note: Most arcs represent many parallel transitions.

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Induced abstractions

Definition (induced abstractions)

Let $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$ be a transition system, and let $\alpha : S \rightarrow S'$ be a surjective function.

The **abstraction (of \mathcal{T}) induced by α** , in symbols \mathcal{T}^α , is the transition system $\mathcal{T}^\alpha = \langle S', L, T', s'_0, S'_\star \rangle$ defined by:

- $T' = \{ \langle \alpha(s), \ell, \alpha(t) \rangle \mid \langle s, \ell, t \rangle \in T \}$
- $s'_0 = \alpha(s_0)$
- $S'_\star = \{ \alpha(s) \mid s \in S_\star \}$

Note: It is easy to see that \mathcal{T}^α is an abstraction of \mathcal{T} . It is the “**smallest**” abstraction of \mathcal{T} with abstraction mapping α .

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Induced abstractions: terminology

Let \mathcal{T} and \mathcal{T}' be transition systems and α be a function such that $\mathcal{T}' = \mathcal{T}^\alpha$ (i. e., \mathcal{T}' is the abstraction of \mathcal{T} induced by α).

- α is called a **strict homomorphism** from \mathcal{T} to \mathcal{T}' , and \mathcal{T}' is called a **strictly homomorphic abstraction** of \mathcal{T} .
- If α is bijective, it is called an **isomorphism** between \mathcal{T} and \mathcal{T}' , and the two transition systems are called **isomorphic**.

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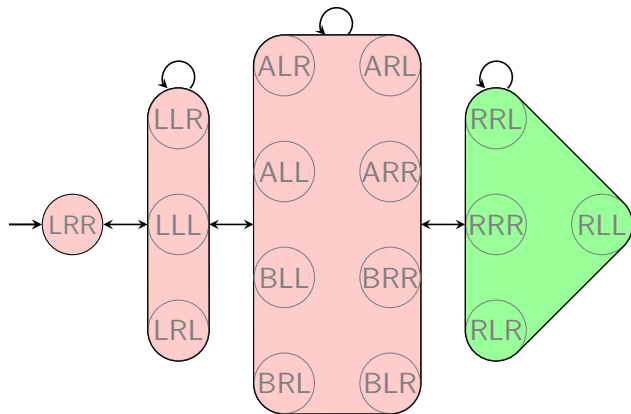
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Strictly homomorphic abstractions: example



This abstraction is a strictly homomorphic abstraction of the concrete transition system \mathcal{T} .

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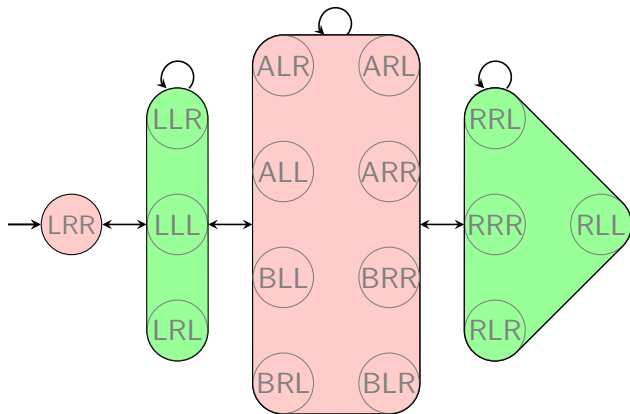
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Strictly homomorphic abstractions: example



If we add any goal states or transitions, it is still an abstraction of \mathcal{T} , but no longer a strictly homomorphic one.

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Abstraction heuristics

Definition (abstraction heuristic induced by an abstraction)

Let Π be an FDR planning task with state space S , and let \mathcal{A} be an abstraction of $\mathcal{T}(\Pi)$ with abstraction mapping α .

The **abstraction heuristic induced by \mathcal{A} and α** , $h^{\mathcal{A},\alpha}$, is the heuristic function $h^{\mathcal{A},\alpha} : S \rightarrow \mathbb{N}_0 \cup \{\infty\}$ which maps each state $s \in S$ to $h_{\mathcal{A}}^*(\alpha(s))$ (the goal distance of $\alpha(s)$ in \mathcal{A}).

Note: $h^{\mathcal{A},\alpha}(s) = \infty$ if no goal state of \mathcal{A} is reachable from $\alpha(s)$

Definition (abstraction heur. induced by strict homomorphism)

Let Π be an FDR planning task and α a strict homomorphism on $\mathcal{T}(\Pi)$. The **abstraction heuristic induced by α** , h^α , is the abstraction heuristic induced by $\mathcal{T}(\Pi)^\alpha$ and α , i. e.,

$$h^\alpha := h^{\mathcal{T}(\Pi)^\alpha, \alpha}.$$

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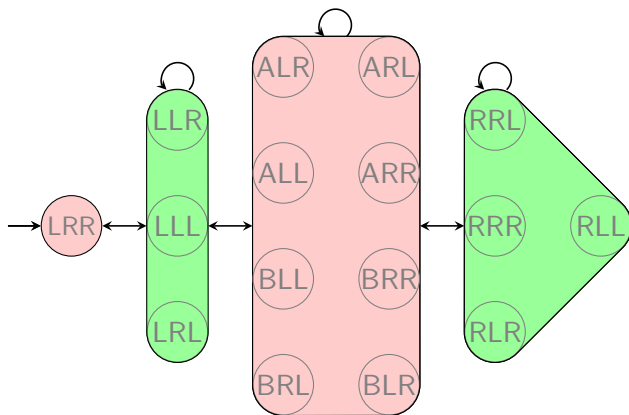
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Abstraction heuristics: example



$$h^{\mathcal{A}, \alpha}(\{p \mapsto L, t_A \mapsto R, t_B \mapsto R\}) = 1$$

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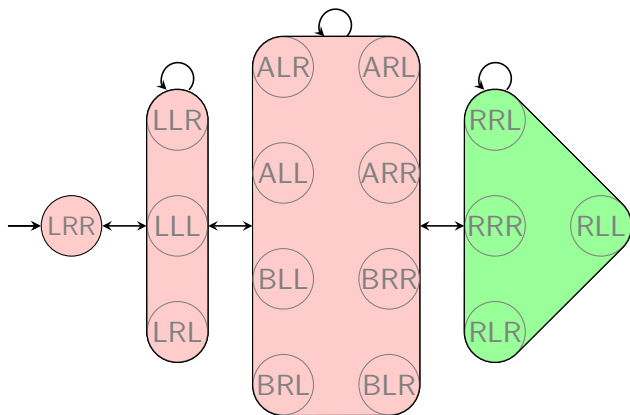
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Abstraction heuristics: example



$$h^\alpha(\{p \mapsto L, t_A \mapsto R, t_B \mapsto R\}) = 3$$

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Consistency of abstraction heuristics

Theorem (consistency and admissibility of $h^{\mathcal{A},\alpha}$)

Let Π be an FDR planning task, and let \mathcal{A} be an abstraction of $\mathcal{T}(\Pi)$ with abstraction mapping α .

Then $h^{\mathcal{A},\alpha}$ is safe, goal-aware, admissible and consistent.

Proof.

We prove goal-awareness and consistency; the other properties follow from these two.

Let $\mathcal{T} = \mathcal{T}(\Pi) = \langle S, L, T, s_0, S_\star \rangle$ and $\mathcal{A} = \langle S', L', T', s'_0, S'_\star \rangle$.

Goal-awareness: We need to show that $h^{\mathcal{A},\alpha}(s) = 0$ for all $s \in S_\star$, so let $s \in S_\star$. Then $\alpha(s) \in S'_\star$ by the definition of abstractions and abstraction mappings, and hence

$$h^{\mathcal{A},\alpha}(s) = h^*_\mathcal{A}(\alpha(s)) = 0.$$

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Consistency of abstraction heuristics

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$$h^{\mathcal{A},\alpha}(s) = h_{\mathcal{A}}^*(\alpha(s)) = 0.$$

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Consistency of abstraction heuristics (ctd.)

Proof (ctd.)

Consistency: Let $s, t \in S$ such that t is a successor of s . We need to prove that $h^{\mathcal{A}, \alpha}(s) \leq h^{\mathcal{A}, \alpha}(t) + 1$.

Since t is a successor of s , there exists an operator o with $app_o(s) = t$ and hence $\langle s, o, t \rangle \in T$.

By the definition of abstractions and abstraction mappings, we get $\langle \alpha(s), o, \alpha(t) \rangle \in T' \rightsquigarrow \alpha(t)$ is a successor of $\alpha(s)$ in \mathcal{A} .

Therefore, $h^{\mathcal{A}, \alpha}(s) = h_{\mathcal{A}}^*(\alpha(s)) \leq h_{\mathcal{A}}^*(\alpha(t)) + 1 = h^{\mathcal{A}, \alpha}(t) + 1$, where the inequality holds because the shortest path from $\alpha(s)$ to the goal in \mathcal{A} cannot be longer than the shortest path from $\alpha(s)$ to the goal via $\alpha(t)$. □

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Consistency of abstraction heuristics (ctd.)

Proof (ctd.)

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Proof (ctd.)

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Orthogonality of abstraction mappings

Definition (orthogonal abstraction mappings)

Let α_1 and α_2 be abstraction mappings on \mathcal{T} .

We say that α_1 and α_2 are **orthogonal** if for all transitions $\langle s, \ell, t \rangle$ of \mathcal{T} , we have $\alpha_i(s) = \alpha_i(t)$ for at least one $i \in \{1, 2\}$.

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Affecting transition labels

Definition (affecting transition labels)

Let \mathcal{T} be a transition system, and let l be one of its labels. We say that l **affects** \mathcal{T} if \mathcal{T} has a transition $\langle s, l, t \rangle$ with $s \neq t$.

Theorem (affecting labels vs. orthogonality)

*Let \mathcal{A}_1 be an abstraction of \mathcal{T} with abstraction mapping α_1 .
Let \mathcal{A}_2 be an abstraction of \mathcal{T} with abstraction mapping α_2 .
If no label of \mathcal{T} affects both \mathcal{A}_1 and \mathcal{A}_2 , then α_1 and α_2 are orthogonal.*

(Easy proof omitted.)

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Orthogonal abstraction mappings: example

	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

Are the abstraction mappings orthogonal?

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Orthogonality and additivity

Theorem (additivity for orthogonal abstraction mappings)

Let $h^{A_1, \alpha_1}, \dots, h^{A_n, \alpha_n}$ be abstraction heuristics for the same planning task Π such that α_i and α_j are orthogonal for all $i \neq j$.

Then $\sum_{i=1}^n h^{A_i, \alpha_i}$ is a safe, goal-aware, admissible and consistent heuristic for Π .

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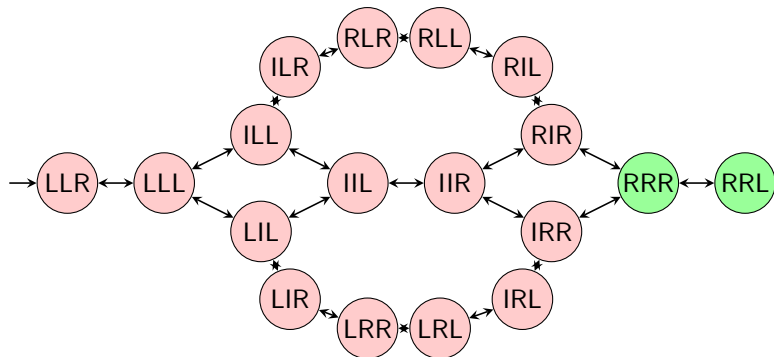
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Orthogonality and additivity: example



transition system \mathcal{T}

state variables: first package, second package, truck

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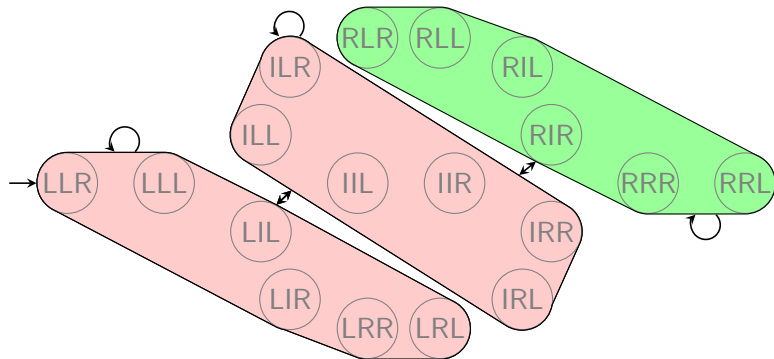
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Orthogonality and additivity: example



abstraction \mathcal{A}_1

mapping: only consider state of first package

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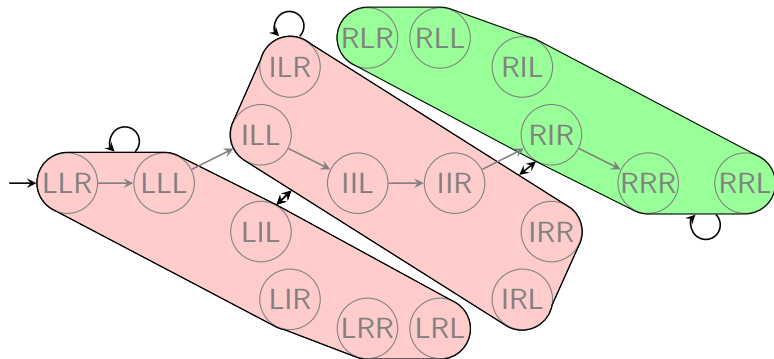
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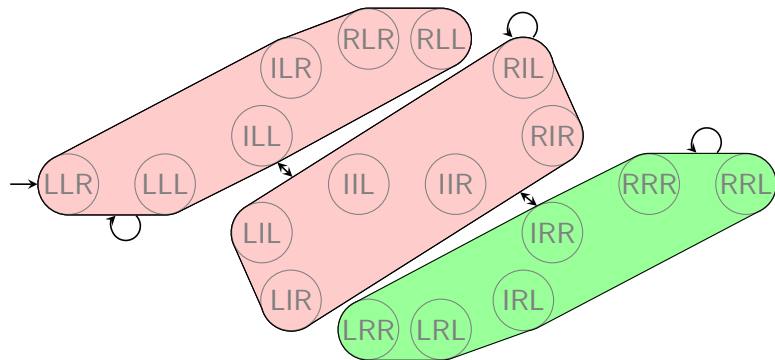
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Orthogonality and additivity: example



abstraction \mathcal{A}_2 (orthogonal to \mathcal{A}_1)
mapping: only consider state of second package

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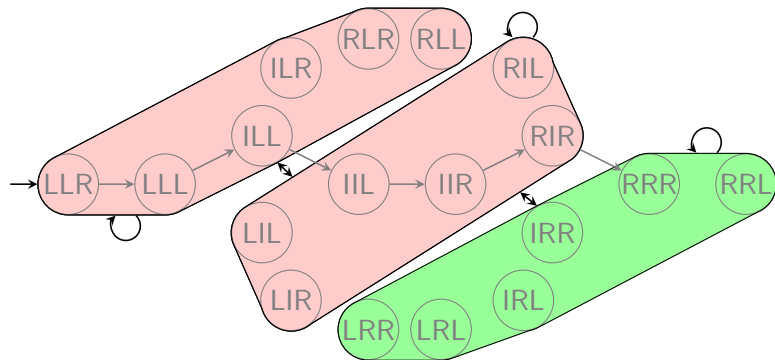
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abstraction \mathcal{A}_2 (orthogonal to \mathcal{A}_1)
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Orthogonality and additivity: proof

Proof.

We prove goal-awareness and consistency;
the other properties follow from these two.

Let $\mathcal{T} = \mathcal{T}(\Pi) = \langle S, L, T, s_0, S_\star \rangle$.

Goal-awareness: For goal states $s \in S_\star$,

$\sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s) = \sum_{i=1}^n 0 = 0$ because all individual
abstractions are goal-aware.

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Summary

Orthogonality and additivity: proof (ctd.)

Proof (ctd.)

Consistency: Let $s, t \in S$ such that t is a successor of s .

Let $L := \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s)$ and $R := \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(t)$.

We need to prove that $L \leq R + 1$.

Since t is a successor of s , there exists an operator o with $app_o(s) = t$ and hence $\langle s, o, t \rangle \in T$.

Because the abstraction mappings are orthogonal, $\alpha_i(s) \neq \alpha_i(t)$ for **at most one** $i \in \{1, \dots, n\}$.

Case 1: $\alpha_i(s) = \alpha_i(t)$ for all $i \in \{1, \dots, n\}$.

$$\begin{aligned} \text{Then } L &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s) \\ &= \sum_{i=1}^n h_{\mathcal{A}_i}^*(\alpha_i(s)) \\ &= \sum_{i=1}^n h_{\mathcal{A}_i}^*(\alpha_i(t)) \\ &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(t) \\ &= R \leq R + 1. \end{aligned}$$

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Orthogonality and additivity: proof (ctd.)

Proof (ctd.)

Case 2: $\alpha_i(s) \neq \alpha_i(t)$ for exactly one $i \in \{1, \dots, n\}$.

Let $k \in \{1, \dots, n\}$ such that $\alpha_k(s) \neq \alpha_k(t)$.

$$\begin{aligned} \text{Then } L &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(s) \\ &= \sum_{i \in \{1, \dots, n\} \setminus \{k\}} h_{\mathcal{A}_i}^*(\alpha_i(s)) + h^{\mathcal{A}_k, \alpha_k}(s) \\ &\leq \sum_{i \in \{1, \dots, n\} \setminus \{k\}} h_{\mathcal{A}_i}^*(\alpha_i(t)) + h^{\mathcal{A}_k, \alpha_k}(t) + 1 \\ &= \sum_{i=1}^n h^{\mathcal{A}_i, \alpha_i}(t) + 1 \\ &= R + 1, \end{aligned}$$

where the inequality holds because $\alpha_i(s) = \alpha_i(t)$ for all $i \neq k$ and $h^{\mathcal{A}_k, \alpha_k}$ is consistent. □

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Abstractions of abstractions

Theorem (transitivity of abstractions)

Let \mathcal{T} , \mathcal{T}' and \mathcal{T}'' be transition systems.

- If \mathcal{T}' is an abstraction of \mathcal{T} and \mathcal{T}'' is an abstraction of \mathcal{T}' , then \mathcal{T}'' is an abstraction of \mathcal{T} .
- If \mathcal{T}' is a strictly homomorphic abstraction of \mathcal{T} and \mathcal{T}'' is a strictly homomorphic abstraction of \mathcal{T}' , then \mathcal{T}'' is a strictly homomorphic abstraction of \mathcal{T} .

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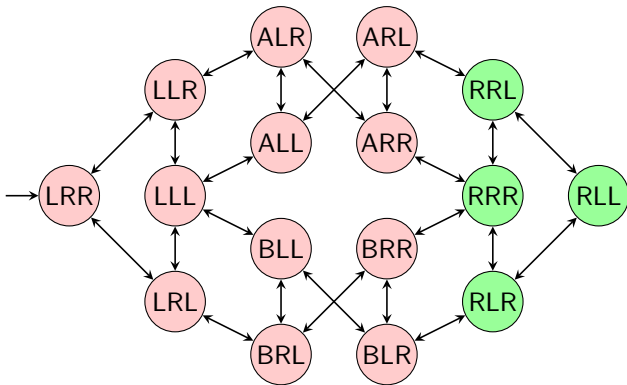
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Abstractions of abstractions: example



transition system \mathcal{T}

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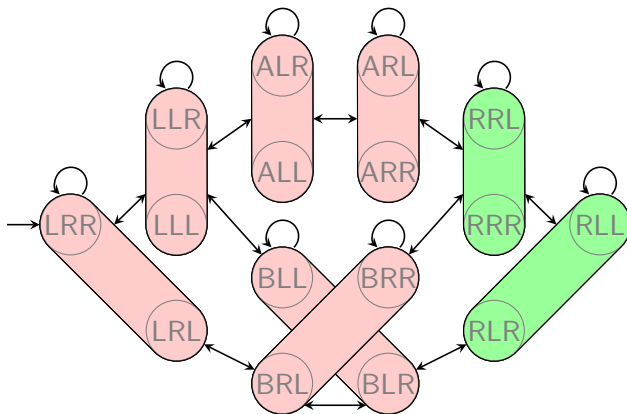
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Abstractions of abstractions: example



Transition system \mathcal{T}' as an abstraction of \mathcal{T}

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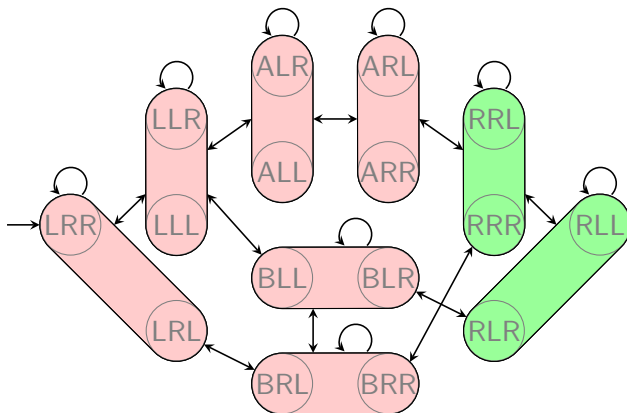
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Abstractions of abstractions: example



Transition system T' as an abstraction of T

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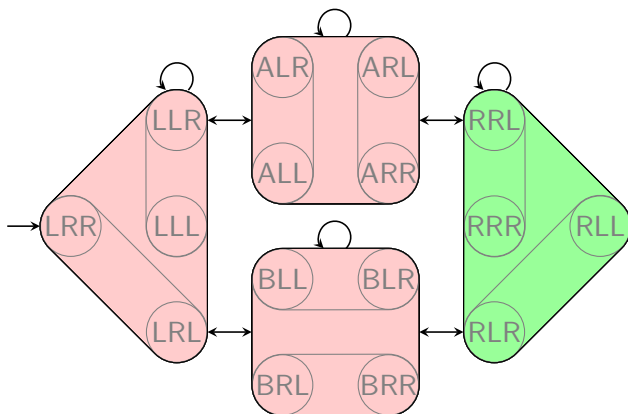
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Transition system \mathcal{T}'' as an abstraction of \mathcal{T}'

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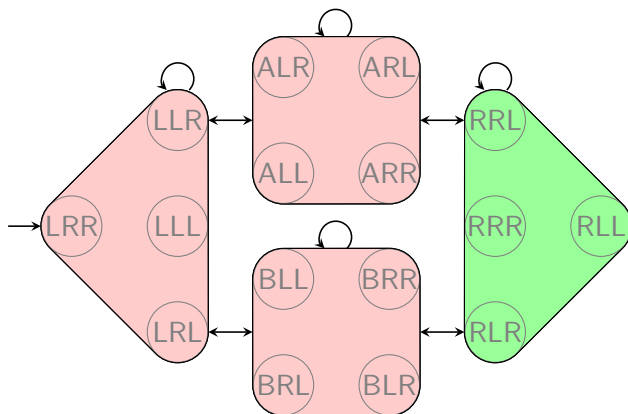
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Summary

Abstractions of abstractions (proof)

Proof.

Let $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$, let $\mathcal{T}' = \langle S', L, T', s'_0, S'_\star \rangle$ be an abstraction of \mathcal{T} with abstraction mapping α , and let $\mathcal{T}'' = \langle S'', L, T'', s''_0, S''_\star \rangle$ be an abstraction of \mathcal{T}' with abstraction mapping α' .

We show that \mathcal{T}'' is an abstraction of \mathcal{T} with abstraction mapping $\beta := \alpha' \circ \alpha$, i. e., that

- 1 $\beta(s_0) = s''_0$,
- 2 for all $s \in S_\star$, we have $\beta(s) \in S''_\star$, and
- 3 for all $\langle s, l, t \rangle \in T$, we have $\langle \beta(s), l, \beta(t) \rangle \in T''$.

Moreover, we show that if α and α' are strict homomorphisms, then β is also a strict homomorphism.

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Abstractions of abstractions: proof

Proof (ctd.)

1. $\beta(s_0) = s''_0$

Because \mathcal{T}' is an abstraction of \mathcal{T} with mapping α , we have $\alpha(s_0) = s'_0$. Because \mathcal{T}'' is an abstraction of \mathcal{T}' with mapping α' , we have $\alpha'(s'_0) = s''_0$.

Hence $\beta(s_0) = \alpha'(\alpha(s_0)) = \alpha'(s'_0) = s''_0$.

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Abstractions of abstractions: proof (ctd.)

Proof (ctd.)

2. For all $s \in S_\star$, we have $\beta(s) \in S_\star''$:

Let $s \in S_\star$. Because \mathcal{T}' is an abstraction of \mathcal{T} with mapping α , we have $\alpha(s) \in S_\star'$. Because \mathcal{T}'' is an abstraction of \mathcal{T}' with mapping α' and $\alpha(s) \in S_\star'$, we have $\alpha'(\alpha(s)) \in S_\star''$.

Hence $\beta(s) = \alpha'(\alpha(s)) \in S_\star''$.

Strict homomorphism if α and α' strict homomorphisms:

Let $s'' \in S_\star''$. Because α' is a strict homomorphism, there exists a state $s' \in S_\star'$ such that $\alpha'(s') = s''$. Because α is a strict homomorphism, there exists a state $s \in S_\star$ such that $\alpha(s) = s'$. Thus $s'' = \alpha'(\alpha(s)) = \beta(s)$ for some $s \in S_\star$.

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Proof (ctd.)

2. For all $s \in S_\star$, we have $\beta(s) \in S''_\star$:

Let $s \in S_\star$. Because \mathcal{T}' is an abstraction of \mathcal{T} with mapping α , we have $\alpha(s) \in S'_\star$. Because \mathcal{T}'' is an abstraction of \mathcal{T}' with mapping α' and $\alpha(s) \in S'_\star$, we have $\alpha'(\alpha(s)) \in S''_\star$.

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Proof (ctd.)

3. For all $\langle s, \ell, t \rangle \in T$, we have $\langle \beta(s), \ell, \beta(t) \rangle \in T''$

Let $\langle s, \ell, t \rangle \in T$. Because \mathcal{T}' is an abstraction of \mathcal{T} with mapping α , we have $\langle \alpha(s), \ell, \alpha(t) \rangle \in T'$. Because \mathcal{T}'' is an abstraction of \mathcal{T}' with mapping α' and $\langle \alpha(s), \ell, \alpha(t) \rangle \in T'$, we have $\langle \alpha'(\alpha(s)), \ell, \alpha'(\alpha(t)) \rangle \in T''$.

Hence $\langle \beta(s), \ell, \beta(t) \rangle = \langle \alpha'(\alpha(s)), \ell, \alpha'(\alpha(t)) \rangle \in T''$.

Strict homomorphism if α and α' strict homomorphisms:

Let $\langle s'', \ell, t'' \rangle \in T''$. Because α' is a strict homomorphism, there exists a transition $\langle s', \ell, t' \rangle \in T'$ such that $\alpha'(s') = s''$ and $\alpha'(t') = t''$. Because α is a strict homomorphism, there exists a transition $\langle s, \ell, t \rangle \in T$ such that $\alpha(s) = s'$ and $\alpha(t) = t'$.

Thus $\langle s'', \ell, t'' \rangle = \langle \alpha'(\alpha(s)), \ell, \alpha'(\alpha(t)) \rangle = \langle \beta(s), \ell, \beta(t) \rangle$ for some $\langle s, \ell, t \rangle \in T$. □

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Coarsenings and refinements

Terminology: Let \mathcal{T} be a transition system, let \mathcal{T}' be an abstraction of \mathcal{T} with abstraction mapping α , and let \mathcal{T}'' be an abstraction of \mathcal{T}' with abstraction mapping α' .

Then:

- $\langle \mathcal{T}'', \alpha' \circ \alpha \rangle$ is called a **coarsening** of $\langle \mathcal{T}', \alpha \rangle$, and
- $\langle \mathcal{T}', \alpha \rangle$ is called a **refinement** of $\langle \mathcal{T}'', \alpha' \circ \alpha \rangle$.

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Heuristic quality of refinements

Theorem (heuristic quality of refinements)

Let $h^{\mathcal{A},\alpha}$ and $h^{\mathcal{B},\beta}$ be abstraction heuristics for the same planning task Π such that $\langle \mathcal{A}, \alpha \rangle$ is a refinement of $\langle \mathcal{B}, \beta \rangle$. Then $h^{\mathcal{A},\alpha}$ dominates $h^{\mathcal{B},\beta}$.

In other words, $h^{\mathcal{A},\alpha}(s) \geq h^{\mathcal{B},\beta}(s)$ for all states s of Π .

Proof.

Since $\langle \mathcal{A}, \alpha \rangle$ is a refinement of $\langle \mathcal{B}, \beta \rangle$, there exists a mapping α' such that $\beta = \alpha' \circ \alpha$ and \mathcal{B} is an abstraction of \mathcal{A} with abstraction mapping α' .

For any state s of Π , we get $h^{\mathcal{B},\beta}(s) = h_{\mathcal{B}}^*(\beta(s)) = h_{\mathcal{B}}^*(\alpha'(\alpha(s))) = h^{\mathcal{B},\alpha'}(\alpha(s)) \leq h_{\mathcal{A}}^*(\alpha(s)) = h^{\mathcal{A},\alpha}(s)$, where the inequality holds because $h^{\mathcal{B},\alpha'}$ is an admissible heuristic in the transition system \mathcal{A} .

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Isomorphic transition systems

Definition (isomorphic transition systems)

Let $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$ and $\mathcal{T}' = \langle S', L', T', s'_0, S'_\star \rangle$ be transition systems.

We say that \mathcal{T} is **isomorphic to** \mathcal{T}' , in symbols $\mathcal{T} \sim \mathcal{T}'$, if there exist bijective functions $\varphi : S \rightarrow S'$ and $\psi : L \rightarrow L'$ such that:

- $\varphi(s_0) = s'_0$,
- $s \in S_\star$ iff $\varphi(s) \in S'_\star$, and
- $\langle s, l, t \rangle \in T$ iff $\langle \varphi(s), \psi(l), \varphi(t) \rangle \in T'$.

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Graph-equivalent transition systems

Definition (graph-equivalent transition systems)

Let $\mathcal{T} = \langle S, L, T, s_0, S_\star \rangle$ and $\mathcal{T}' = \langle S', L', T', s'_0, S'_\star \rangle$ be transition systems.

We say that \mathcal{T} is graph-equivalent to \mathcal{T}' , in symbols $\mathcal{T} \cong \mathcal{T}'$, if there exists a bijective function $\varphi : S \rightarrow S'$ such that:

- $\varphi(s_0) = s'_0$,
- $s \in S_\star$ iff $\varphi(s) \in S'_\star$, and
- $\langle s, \ell, t \rangle \in T$ for some $\ell \in L$ iff $\langle \varphi(s), \ell', \varphi(t) \rangle \in T'$ for some $\ell' \in L'$.

Note: There is no requirement that the labels of \mathcal{T} and \mathcal{T}' correspond in any way. For example, it is permitted that all transitions of \mathcal{T} have different labels and all transitions of \mathcal{T}' have the same label.

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Isomorphism vs. graph equivalence

- (\sim) and $(\overset{G}{\sim})$ are equivalence relations.
- Two isomorphic transition systems are interchangeable for all practical intents and purposes.
- Two graph-equivalent transition systems are interchangeable for most intents and purposes. In particular, their state distances are identical, so they define the same abstraction heuristic for corresponding abstraction functions.
- Isomorphism implies graph equivalence, but not vice versa.

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Using abstraction heuristics in practice

In practice, there are conflicting goals for abstractions:

- we want to obtain an **informative heuristic**, but
- want to keep its **representation small**.

Abstractions have small representations if they have

- **few abstract states** and
- a **succinct encoding** for α .

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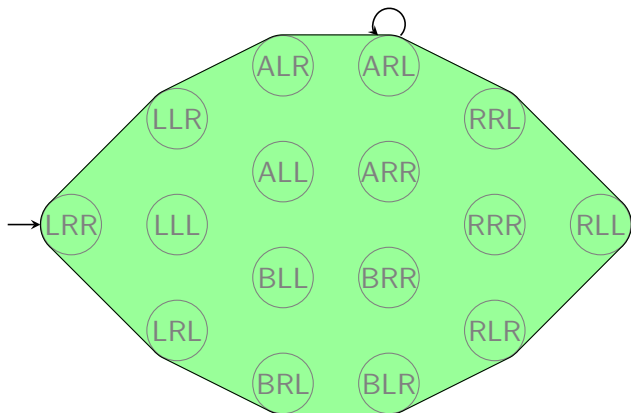
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Counterexample: one-state abstraction



One-state abstraction: $\alpha(s) := \text{const.}$

- + very few abstract states and succinct encoding for α
- completely uninformative heuristic

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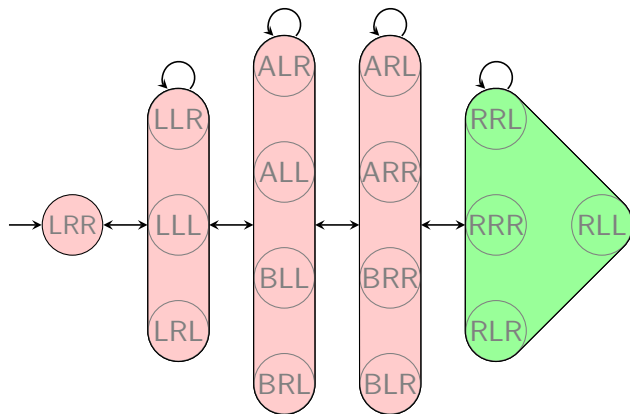
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Counterexample: perfect abstraction



Perfect abstraction: $\alpha(s) := h^*(s)$.

- + perfect heuristic and usually few abstract states
- usually no succinct encoding for α

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Automatically deriving good abstraction heuristics

Abstraction heuristics for planning: main research problem

Automatically derive effective abstraction heuristics
for planning tasks.

⇒ we will study two state-of-the-art approaches
in the next two chapters

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- An **abstraction** relates a transition system \mathcal{T} (e. g. of a planning task) to another (usually smaller) transition system \mathcal{T}' via an **abstraction mapping** α .
- Abstraction **preserves all important aspects** of \mathcal{T} : initial state, goal states and (labeled) transitions.
- Hence, they can be used to define **heuristics** for the original system \mathcal{T} : estimate the goal distance of s in \mathcal{T} by the optimal goal distance of $\alpha(s)$ in \mathcal{T}' .
- Such **abstraction heuristics** are **safe**, **goal-aware**, **admissible** and **consistent**.

Summary (ctd.)

- **Strictly homomorphic abstractions** are desirable as they do not include “unnecessary” abstract goal states or transitions (which could lower heuristic values).
- Any surjection from the states of \mathcal{T} to any set induces a strictly homomorphic abstraction in a natural way.
- Multiple abstraction heuristics can be summed without losing properties like admissibility if the underlying abstraction mapping are **orthogonal**.
- One sufficient condition for orthogonality is that abstractions are **affected** by disjoint sets of labels.

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Summary (ctd.)

- The process of abstraction is **transitive**: an abstraction can be abstracted further to yield another abstraction.
- Based on this notion, we can define abstractions that are **coarsenings** or **refinements** of others.
- A refinement can never lead to a worse heuristic.
- Practically useful abstractions are those which give **informative heuristics**, yet have a **small representation**.

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