Grammars

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Overview

- Sipser: the automata/machine approach
- Schoening: the grammar approach
- State links among the two approaches
- See more types of grammars:
  - Regular grammars
  - Context-free
  - Context-sensitive
  - Grammar
- Based on: Uwe Schoening's "Theoretische Informatik – Kurzgefasst", Spektrum.

Grammars

DEFINITION:

A **grammar** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the **variables,**
2. Σ is a finite set, disjoint from V, called the **terminals**
3. R is a finite set of **rules**, with each rule $u \to v$ having $u \in (V \cup \Sigma)^+$ and $v \in (V \cup \Sigma)^*$
4. $S \in V$ is the **start symbol**.

Most concepts carry over from CFGs, i.e. derivation, language accepted by grammar, ambiguity, leftmost derivation, …

Natural language example:

⟨SENTENCE⟩	→	⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
⟨NOUN-PHRASE⟩	→	⟨CMPLX-NOUN⟩|⟨CMPLX-NOUN⟩⟨PREP-PHRASE⟩
⟨VERB-PHRASE⟩	→	⟨CMPLX-VERB⟩|⟨CMPLX-VERB⟩⟨PREP-PHRASE⟩
⟨PREP-PHRASE⟩	→	⟨PREP⟩⟨CMPLX-NOUN⟩
⟨CMPLX-NOUN⟩	→	⟨ARTICLE⟩⟨NOUN⟩
⟨CMPLX-VERB⟩	→	⟨VERB⟩|⟨VERB⟩⟨NOUN-PHRASE⟩
⟨ARTICLE⟩	→	a | the
⟨NOUN⟩	→	boy | girl | flower
⟨VERB⟩	→	touches | likes | sees
⟨PREP⟩	→	with

a boy sees

the boy sees a flower

a girl with a flower likes the boy

⟨SENTENCE⟩ → ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩

⟨NOUN-PHRASE⟩ → ⟨CMPLX-NOUN⟩|⟨CMPLX-NOUN⟩⟨PREP-PHRASE⟩

⟨VERB-PHRASE⟩ → ⟨CMPLX-VERB⟩|⟨CMPLX-VERB⟩⟨PREP-PHRASE⟩

⟨PREP-PHRASE⟩ → ⟨PREP⟩⟨CMPLX-NOUN⟩

⟨CMPLX-NOUN⟩ → ⟨ARTICLE⟩⟨NOUN⟩

⟨CMPLX-VERB⟩ → ⟨VERB⟩|⟨VERB⟩⟨NOUN-PHRASE⟩

⟨ARTICLE⟩ → a | the

⟨NOUN⟩ → boy | girl | flower

⟨VERB⟩ → touches | likes | sees

⟨PREP⟩ → with

⟨SENTENCE⟩ ⇒ ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩

⇒ ⟨CMPLX-NOUN⟩⟨VERB-PHRASE⟩

⇒ ⟨ARTICLE⟩⟨NOUN⟩⟨VERB-PHRASE⟩

⇒ a ⟨NOUN⟩⟨VERB-PHRASE⟩

⇒ a boy ⟨VERB-PHRASE⟩

⇒ a boy ⟨CMPLX-VERB⟩

⇒ a boy ⟨VERB⟩

⇒ a boy sees

Parsing

$G_3 = (V, \Sigma, R, < Expr >\}$

$V = \{< Expr >, < Term >, < Factor >\}$

$\Sigma = \{a, +, \times, (,)\}$

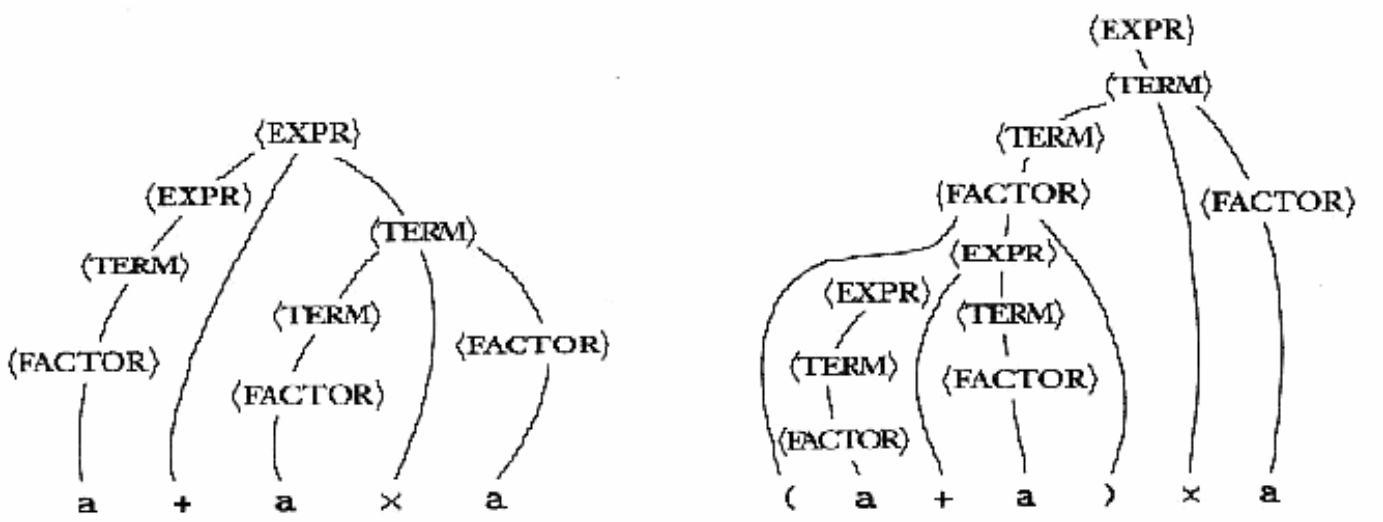
R is

$< Expr > \rightarrow < Expr > + < Term > | < Term >$

$< Term > \rightarrow < Term > \times < Factor > | < Factor >$

$< Factor > \rightarrow (< Expr >) \mid a$

- Construct meaning (parse tree)



- Parse trees for the strings **a + a x a** and **(a + a) x a**

$$S \to aSBC$$
$$S \to aBC$$
$$CB \to BC$$
$$aB \to ab$$
$$bB \to bb$$
$$bC \to bc$$
$$cC \to cc$$

$$L(G) = \{a^n b^n c^n \mid n \geq 1\}$$

$$S \Rightarrow aSBC$$
$$\Rightarrow aaSBCBC$$
$$\Rightarrow aaaBCBCBC$$
$$\Rightarrow aaaBBCCBC$$
$$\Rightarrow aaaBBCBCC$$
$$\Rightarrow aaaBBBCCC$$
$$\Rightarrow aaabBBCCC$$
$$\Rightarrow aaabbBCCC$$
$$\Rightarrow aaabbbCCC$$
$$\Rightarrow aaabbbcCC$$
$$\Rightarrow aaabbbccC$$
$$\Rightarrow aaabbbccc$$

Chomsky Hierarchy

- Type 0 : every grammar; Turing recognisable language
- Type 1 : context-sensitive

  for all rules $u \to v$ holds : $|u| \leq |v|$
- Type 2 : context free

  for all rules $u \to v$ holds : u is a single variable
- Type 3 : regular

  for all rules $u \to v$ holds : u is a single variable

  and v is either a terminal or a terminal followed by a variable
- Exception for

ε : the empty string

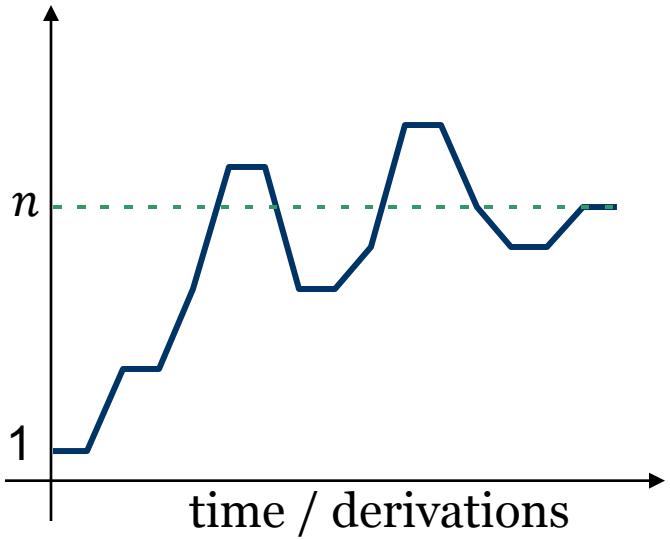
Type 1 : $S \to \varepsilon$ and S start symbol that does not appear at the right hand side

Type 2 and 3 : $A \to \varepsilon$ is allowed

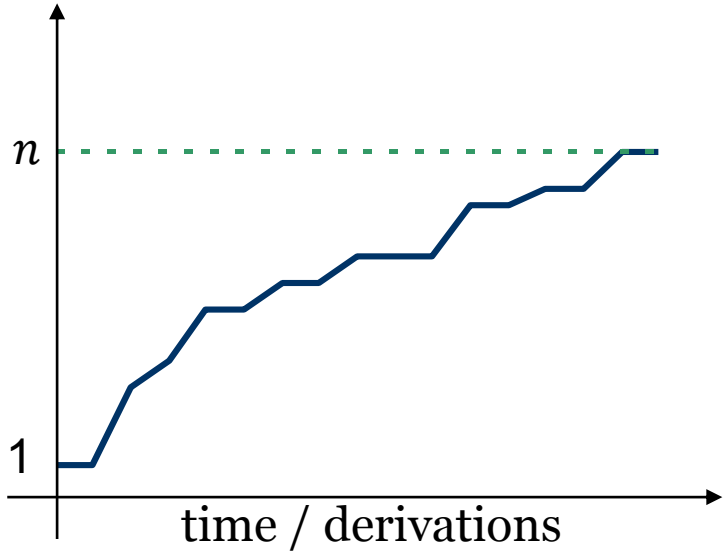
Difference Type 0 and Type 1

- Type 0 : every grammar; Turing recognisable language
- Type 1 : context-sensitive; for all rules $u \rightarrow v$ holds: $|u| \leq |v|$

→ n is the length of your string, then:



Type 0



Type 1

Acceptance Problem for Grammars of Type 1, 2, or 3

$A_{L(G)} = \{(G, w) \mid w \in L(G)\}$

Theorem

$A_{L(G)}$ is decidable if G of Type 1,2 or 3

Proof

For $m, n \in \mathrm{N}$,define

$T_m^n = \{w \in (V \cup \Sigma)^* \mid |w| \leq n$ and w can be derived from S in at most m derivation steps$\}$

$T_m^n, n \geq 1$ can be defined inductively

$T_0^n = \{S\}$

$T_{m+1}^n = Der_n(T_m^n)$ where $Der_n(X) = X \cup \{w \in (V \cup \Sigma)^* \mid |w| \leq n$ and $w' \Rightarrow w$ for some $w' \in X\}$

As there are only a finite number of words

of length n, it must be that for some m

$T_m^n = T_{m+1}^n = T_{m+2}^n = \ldots$

and this T_m^n must contain w if $|\mathrm{w}| = n$ and $w \in L(G)$

Acceptance Problem for Grammars of Type 1, 2, or 3 (ctd.)

Algorithm

Input (G, w); $|w| = n$

$T := \{S\}$

Repeat

$T_1 := T$;

$T := Der_n(T_1)$;

Until $w \in T$ or $T = T_1$

If $w \in T$ then output *yes*; otherwise output *no*

Compute e.g. T_4^4

for the following grammar:

$S \to AB$

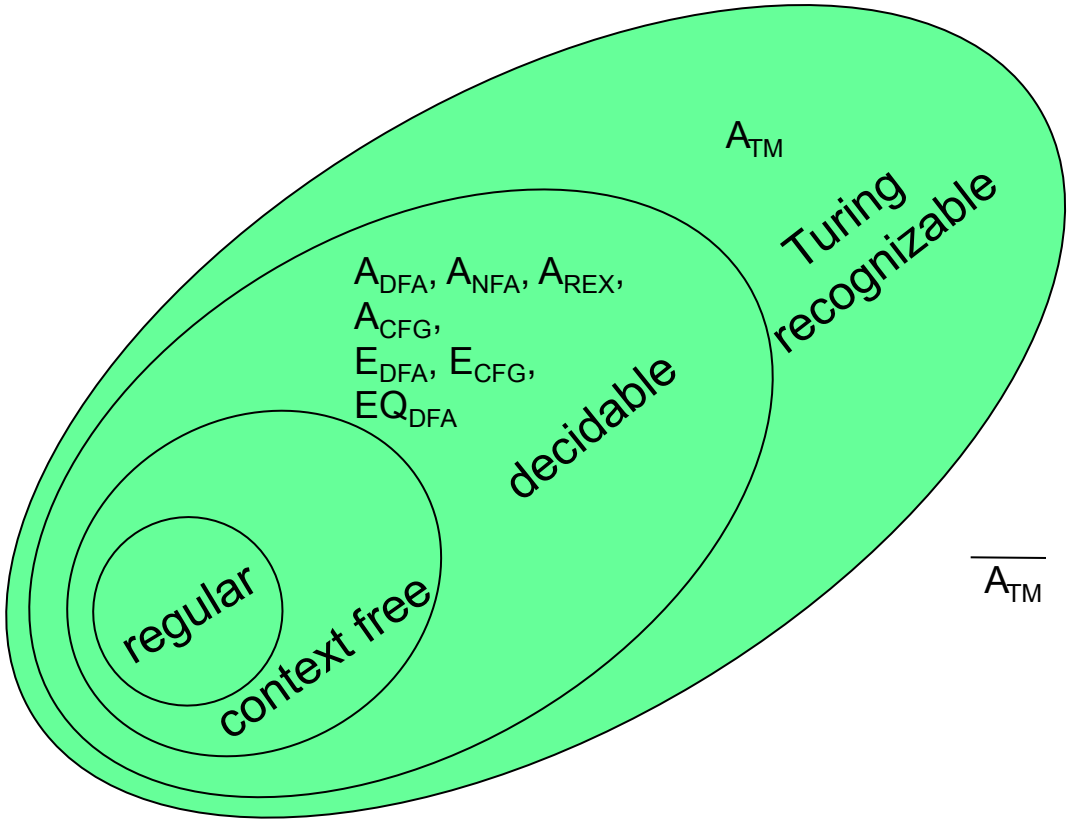
$A \to AB \mid a$

$B \to b$

$T_0^4 = \{S\}$
$T_1^4 = \{S,AB\}$
$T_2^4 = \{S,AB,ABB,aB,Ab\}$
$T_3^4 = \{S,AB,ABB,aB,Ab,ABBB,aBB,ab,ABb,AbB\}$
$T_4^4 = \{S,AB,ABB,aB,Ab,ABBB,aBB,ab,ABb,AbB,$
$aBBB,AbBB,ABbB,ABBb,Abb\}$

$$Type3 \subset Type2 \subset Type1 \subset Decidable \subset Type0 \subset Languages$$

Earlier



The relationship amoung languages

Machines corresponding to languages

- Type 3, regular languages :
  - Regular grammar, DFA, NFA, regular expression
- Type 2, context-free languages :
  - Context free grammar, PDA
- Type 1, context-sensitive language
  - Context sensitive grammar, LBA
- Type 0, Turing recognizable
  - Grammar, Turing machine

Deterministic versus non-deterministic machines

- NFA and DFA are equivalent
- PDA and DPA are not equivalent
  - DPA : deterministic subset of PDA
- The LBA-problem: Are LBA and DLBA equivalent?
  - In short: LBA = DLBA?
  - Still an open question!
- NTM and DTM are equivalent

Non-determ. automata	Deterministic automata	Equivalent?
NFA	DFA	Yes
PDA	DPDA	No
LBA	DLBA	?
TM	DTM	Yes

Closure properties

closed under ?	$\cap$	$\cup$	$\overline{}$	$\times$	*
Regular	yes	yes	yes	yes	yes
Context free	no	yes	no	yes	yes
Context sensitive	yes	yes	yes	yes	yes
Type 0	yes	yes	no	yes	yes

Decidability

decidable ?	*A*	*E*	*EQ*
regular	yes	yes	yes
context free	yes	yes	no
context sensitive	yes	no	no
turing recognizable	no	no	no