# Regular Languages

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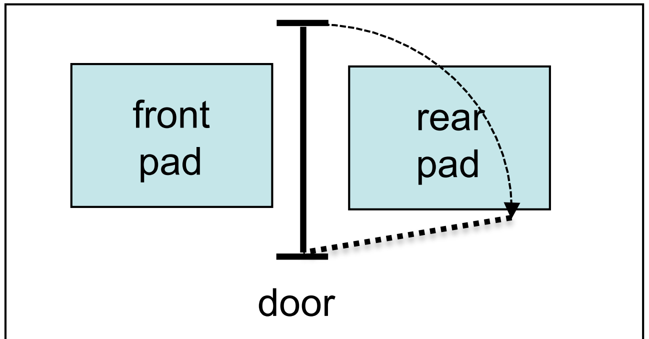
# Overview

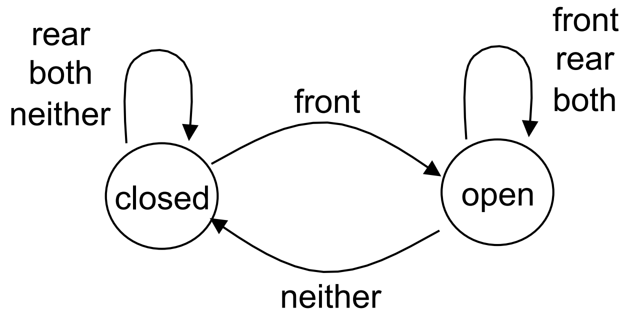
- Deterministic finite automata
- Regular languages
- Nondeterministic finite automata
- Closure operations
- Regular expressions
- Nonregular languages
- The pumping lemma

# Finite automata

- An intuitive example: supermarket door controller




Top view of an automatic door




State diagram for the automatic door controler

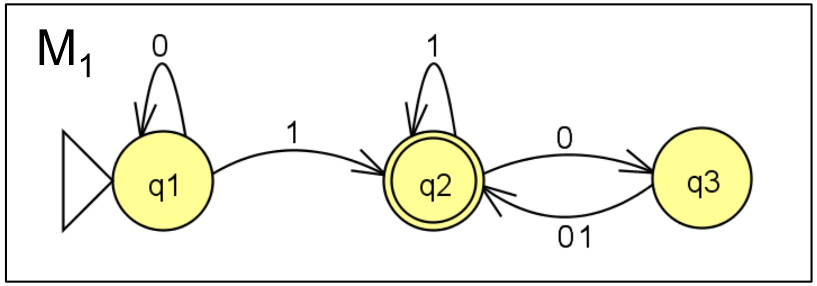
- Probabilistic counterparts exist
  - Markov chains, Bayesian nets, etc.
  - Not in this course

Transition table for the automatic door controler:

| | neither | front | rear | both |
|---|---|---|---|---|
| closed | closed | open | closed | closed |
| open | closed | open | open | open |

# Finite automata (ctd.)

- Example $M_1$ (figure 1.4)




(plotted with JFLAP: www.jflap.org)

states: $q_1, q_2, q_3$

start state: $q_1$

acceptance state: $q_2$

transitions

output: *accept* or *reject*

DEFINITION 1.5:

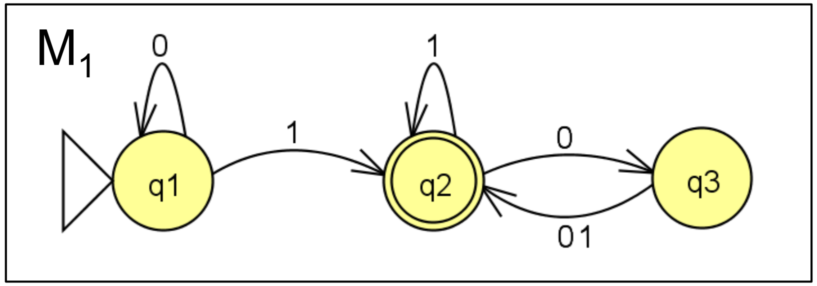
A **finite automaton** M is a 5-tuple

$$M = (Q, \Sigma, \delta, q_0, F)$$

where,

1. $Q$ is a finite set called the **states**
2. $\Sigma$ is a finite set called the **alphabet**
3. $\delta: Q \times \Sigma \rightarrow Q$ is the **transition function**
4. $q_0 \in Q$ is the **start state**
5. $F \subseteq Q$ is the set of **accept states** (also called **final states**)

# Finite automata: example




Which kind of input does $M_1$ accept:

1. „abbbaaa" ?
2. „000000" ?
3. the empty string $\varepsilon$ ?
4. „1000111" ?

Describe $M_1$:

1. $Q = \{q_1, q_2, q_3\}$
2. $\Sigma = \{0, 1\}$
3. $\delta$ defined by transition table:

| | 0 | 1 |
|---|---|---|
| $q_1$ | $q_1$ | $q_2$ |
| $q_2$ | $q_3$ | $q_2$ |
| $q_3$ | $q_2$ | $q_2$ |

4. $q_1$ start state
5. $F = \{q_2\}$

→ Which **language** is accepted by $M_1$ ?

# Finite automaton $M_1$ and language A

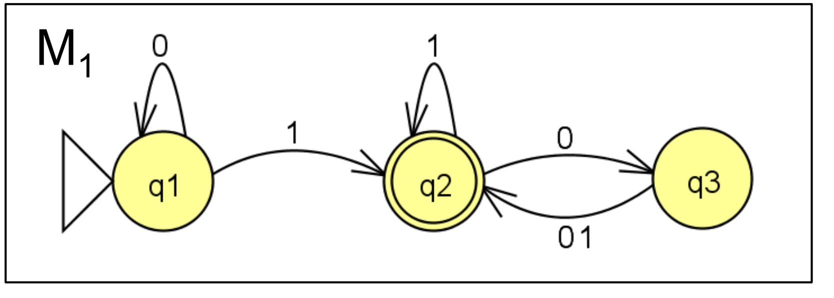
- Let $A$ be the set of strings that a machine $M$ accepts, then
  - "$M$ recognizes $A$"
  - $A$ is the language $L(M)$
- In case of $M_1$, let

$$A = \{w \mid w \ contains \ at \ least \ one \ 1 \ and \ an \ even \ number \ of \ 0s \ follow \ the \ last \ 1\}.$$

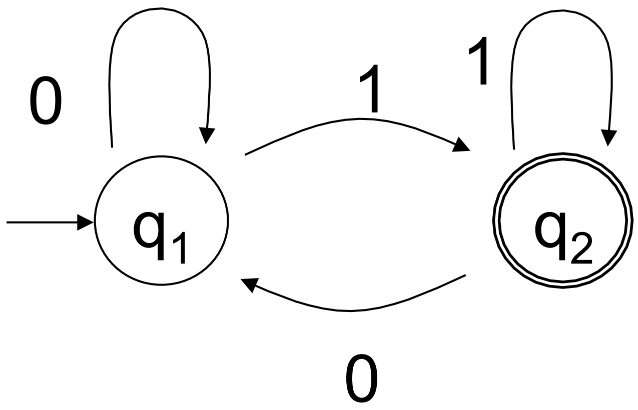
then

$L(M_1) = A$, or equivalently,

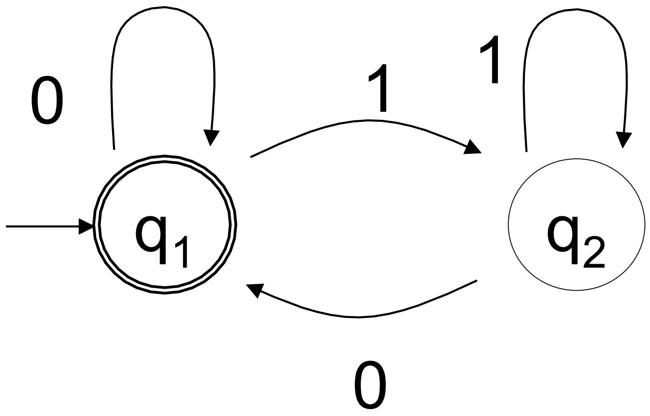
$M_1$ recognizes $A$.



# Finite automata $M_2$ and $M_3$

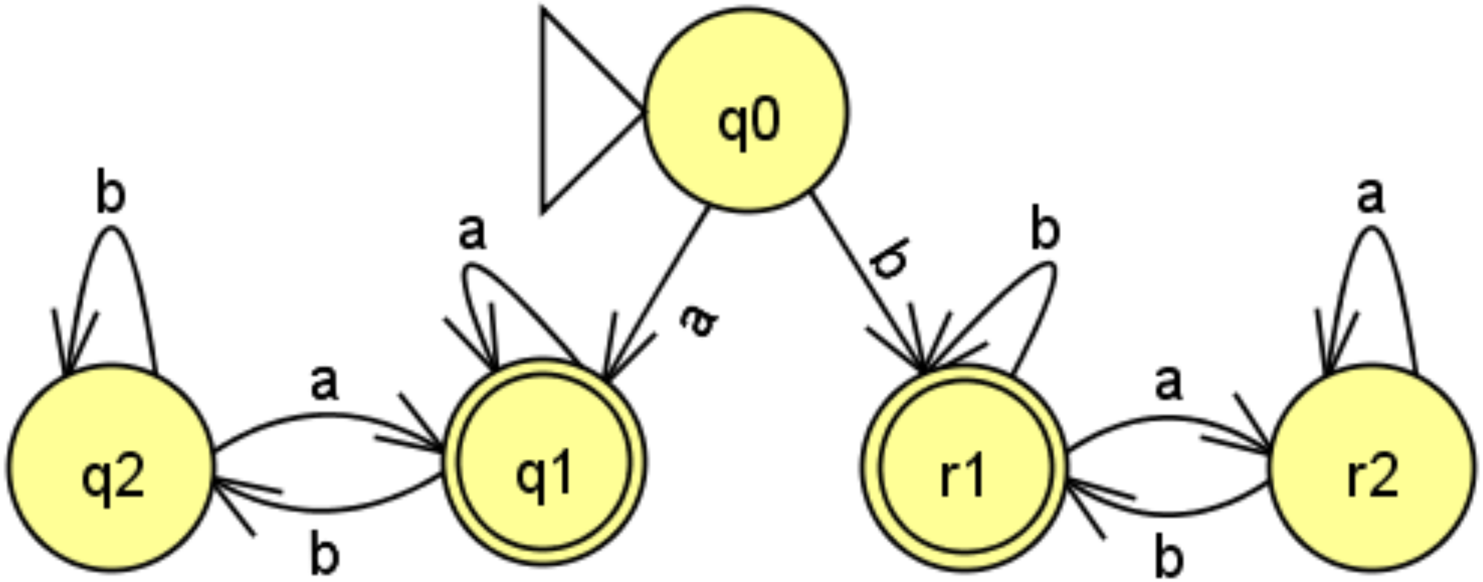



State diagram of the two-state finite automaton $M_2$



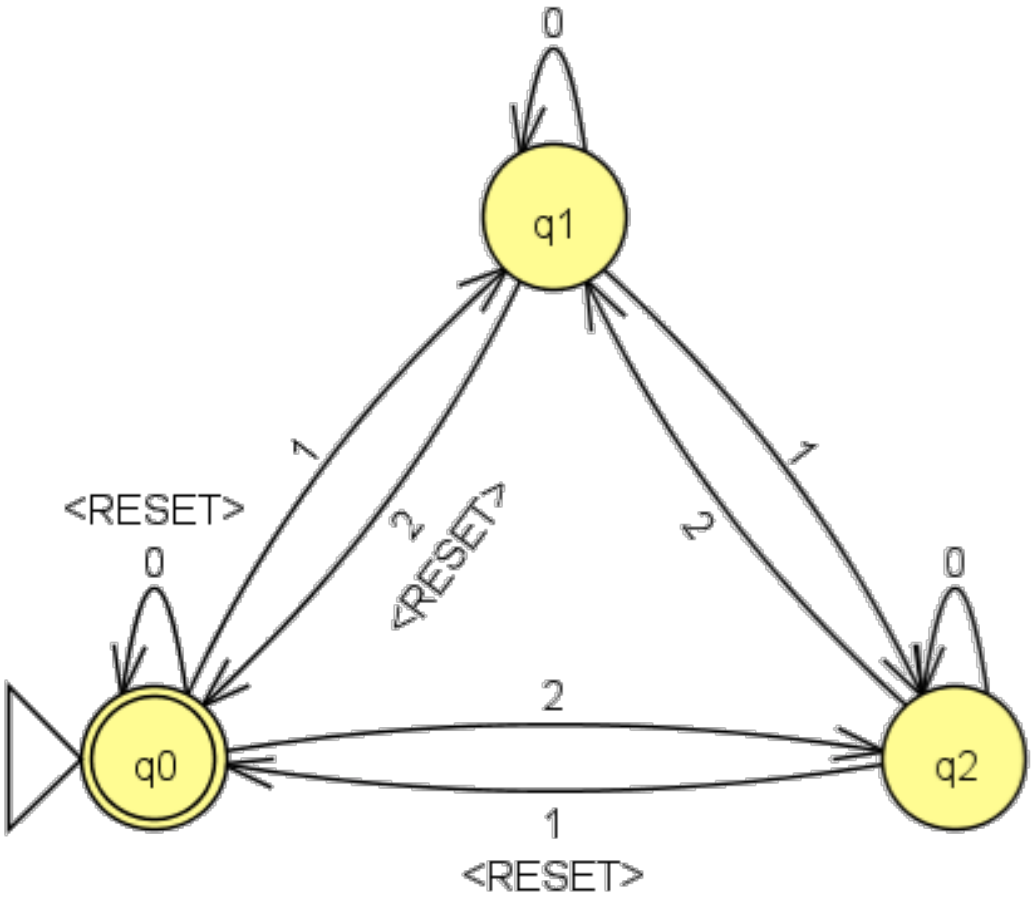

State diagram of the two-state finite automaton $M_3$

# Finite automaton $M_4$




Finite automaton $M_4$ (figure 1.12)

# Finite automaton $M_5$




Finite automaton $M_5$ (figure 1.14)

$M_5$:

- keeps a running count of the sum of all numerical input symbols of its alphabet $\Sigma = \{0,1,2, RESET\}$ that it reads, modulo 3.
- resets the count, every time it receives <RESET>.
- accepts, if the sum is a multiple of 3.

# Formal definition of computation

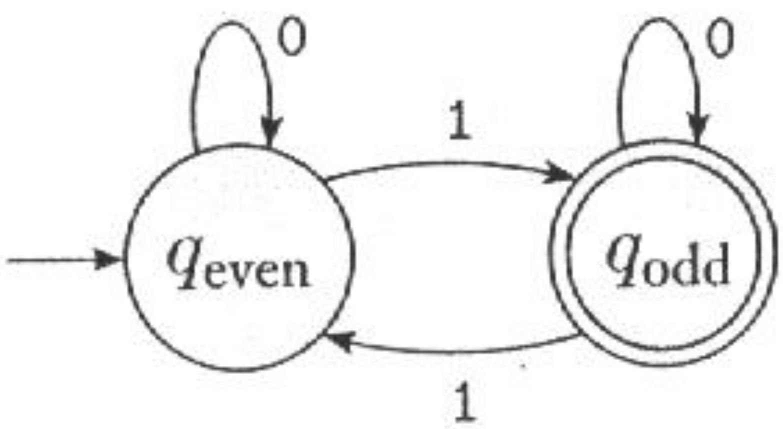
- Let $M$ be a finite automaton $M = (Q, \Sigma, \delta, q_0, F)$
- Let $w = w_1 \dots w_n$ be a string over $\Sigma$
  - $M$ accepts $w$ if a sequence of states $r_0, \dots\ r_n$ exists in $Q$ such that
    1. $r_0 = q_0$
    2. $\delta(r_1, w_{i+1}) = r_{i+1}$ for all $i = 0, \dots, n-1$
    3. $r_n \in F$
  - $M$ recognizes language $A$ if $A = \{w \mid M\ accepts\ w\}$

DEFINITION 1.16:

A language is called **regular language** if some finite automaton recognizes it.

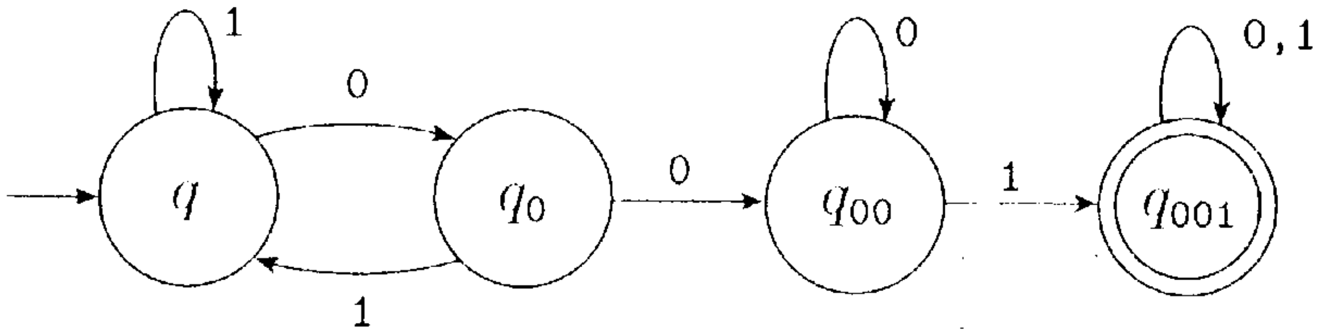
# Designing finite automata

- Design automaton for language consisting of binary strings with an odd number of 1s
- Design first states
- Then transitions
- Start state and accept states



# Another example

- Design an automaton to recognize the language of binary strings containing the string 001 as substring
- We have four possibilities:
  1. we haven‘t seen any symbol of the pattern yet, or
  2. we have seen a 0, or
  3. we have seen a 00, or
  4. we have seen the pattern 001



# The regular operations

- Let $A$ and $B$ be languages.
- We define:
  - **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
  - **Concatenation**: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
  - **Star**: $A^* = \{x_1 x_2 \ldots x_n \mid n \geq 0 \text{ and each } x_i \in A\}$
    - **note** that also $\varepsilon \in A$
- Example: $A = \{empty, full\}; B = \{cup, bottle\}$
- $A \cup B = \ \ldots$
- $A \circ B = \ \ldots$
- $A^* = \ \ldots$

# Regular languages are closed under …

A set $S$ is **closed** under an operation $o$ if applying $o$ on elements of $S$ yields elements of $S$.

- example: multiplication on natural numbers
- counterexample: division of natural numbers

Theorem 1.25:

The class of regular languages is closed under the union operation.
(In other words: If $A_1$ and $A_2$ are regular languages, so is $A_1 \cup A_2$.)

# Proof 1.25 (by construction)

Let $M_1$ recognize $A_1$ where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, and
$M_2$ recognize $A_2$ where $M_1 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct $M$ to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$.

1. $Q = \{(r_1, r_2) | \; r_1 \in Q_1 \; and \; r_2 \in Q_2\}$.
   This set is the **cartesian product** of the sets $Q_1$ and $Q_2$ (written $Q_1 \times Q_2$). It is the set of all pairs of states with the first from $Q_1$ and the second from $Q_2$.
2. $\Sigma$, the alphabet, is the same as in case of $M_1$ and $M_2$. The theorem remains true if they have different alphabets, $\Sigma_1$ and $\Sigma_2$. We would then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$.

# Proof 1.25 (by construction, ctd.)

3. $\delta$, the transistion function, is defined as follows.
   For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let
   $$\delta\big((r_1, r_2), a\big) = (\delta_1(r_1, a), \delta_2(r_2, a)).$$
   Hence $\delta$ gets a state of $M$ (which actually is a pair of states from $M_1$ and $M_2$), together with an input symbol, and returns $M$'s next state.
4. $q_0$ is the pair $(q_1, q_2)$.
5. $F$ is the set of pairs, in which at leadt one member is an accept state of either $M_1$ or $M_2$. We can write this as
   $$F = \{(r_1, r_2) |\, r_1 \in F_1 \; or \; r_2 \in F_2\}.$$
   This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$.
   (Note: it is not the same as $F = F_1 \times F_2$. What would that give us?)

$\blacksquare$

# Example

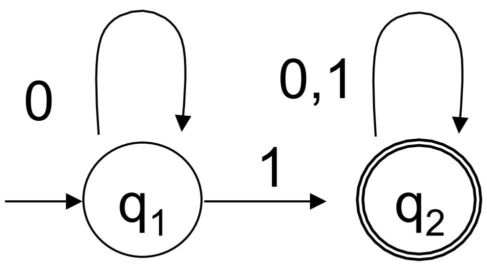
$M = (Q, \Sigma, \delta, q, F)$

constructed from $M_1 = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma_2, \delta_2, q_2, F_2)$

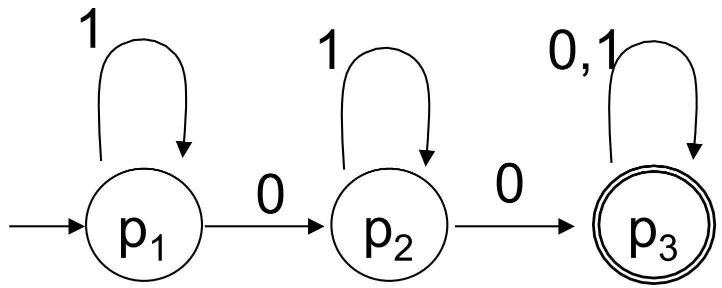
Define

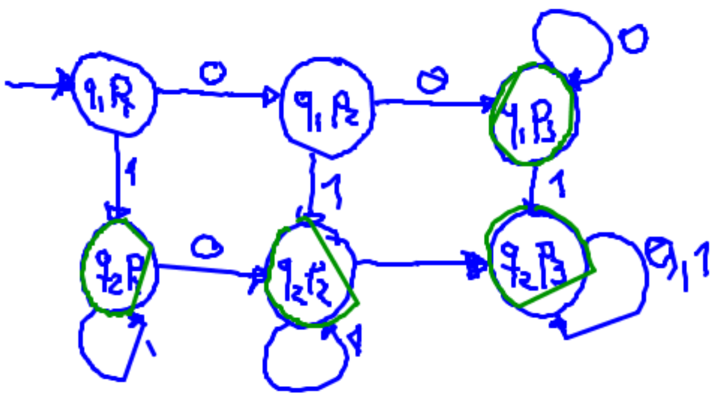
1. $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$
2. $\Sigma = \Sigma_1 \cup \Sigma_2$
3. $\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$
4. $q = (q_1, q_2)$
5. $F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}$

$M_1$ with $L(M_1) = \{w \mid w \text{ contains a } 1\}$




$M_2$ with $L(M_2) = \{w \mid w \text{ contains at least two } 0s\}$



# Regular languages are closed under …

Theorem 1.26:

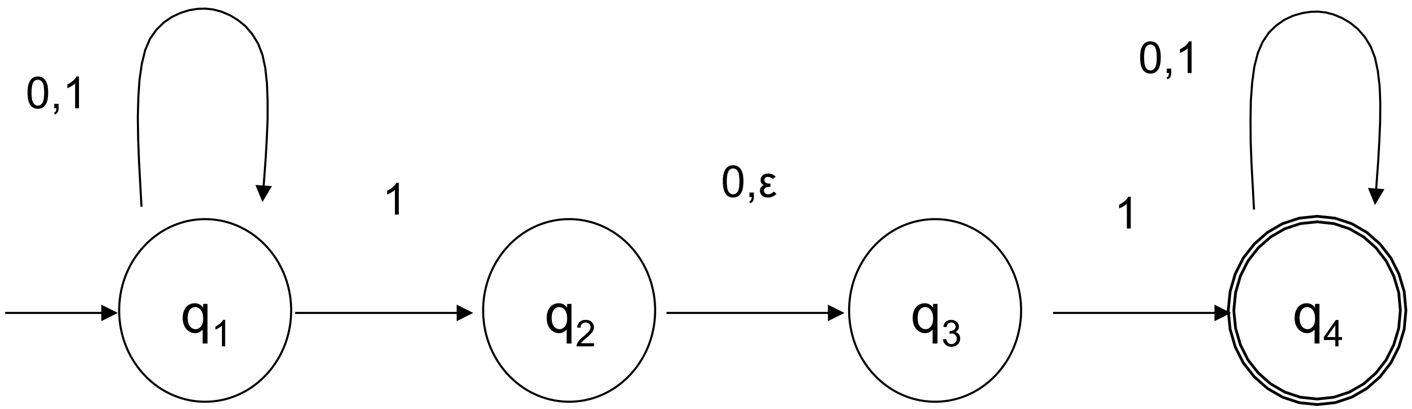
The class of regular languages is closed under the concatenation operation.
(In other words: If $A_1$ and $A_2$ are regular languages, so is $A_1 \circ A_2$.)

⇩

Non deterministic finite automata

# Non deterministic finite automata (NFA)

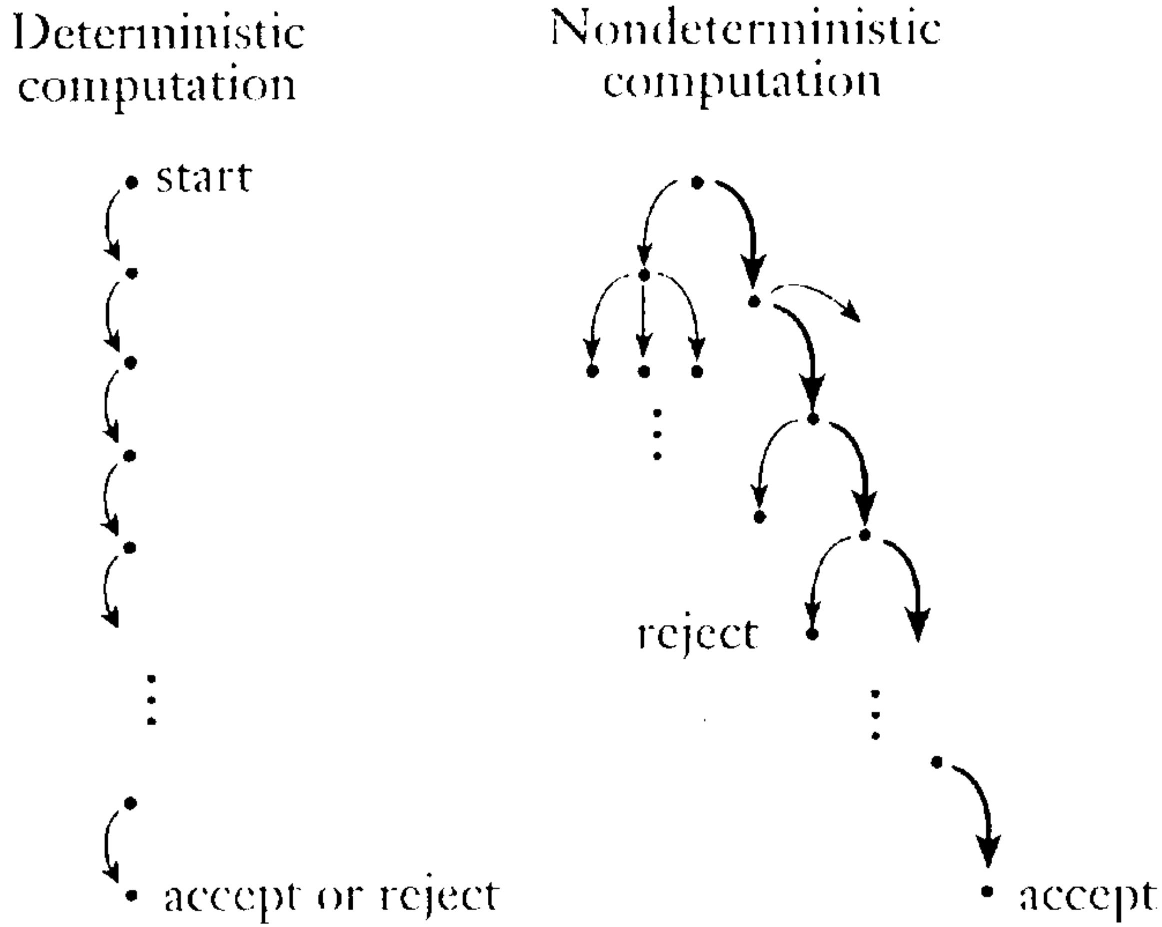
- Deterministic (DFA)
  - *One* successor state
  - ε transitions not allowed
- Non deterministic (NFA)
  - *Several* successor states possible
  - ε transitions possible



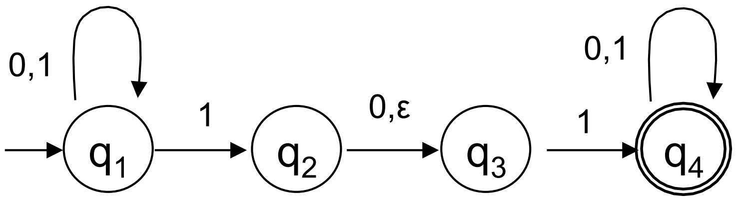

NFA $N_1$

# Deterministic vs. non deterministic computation

Figure 15

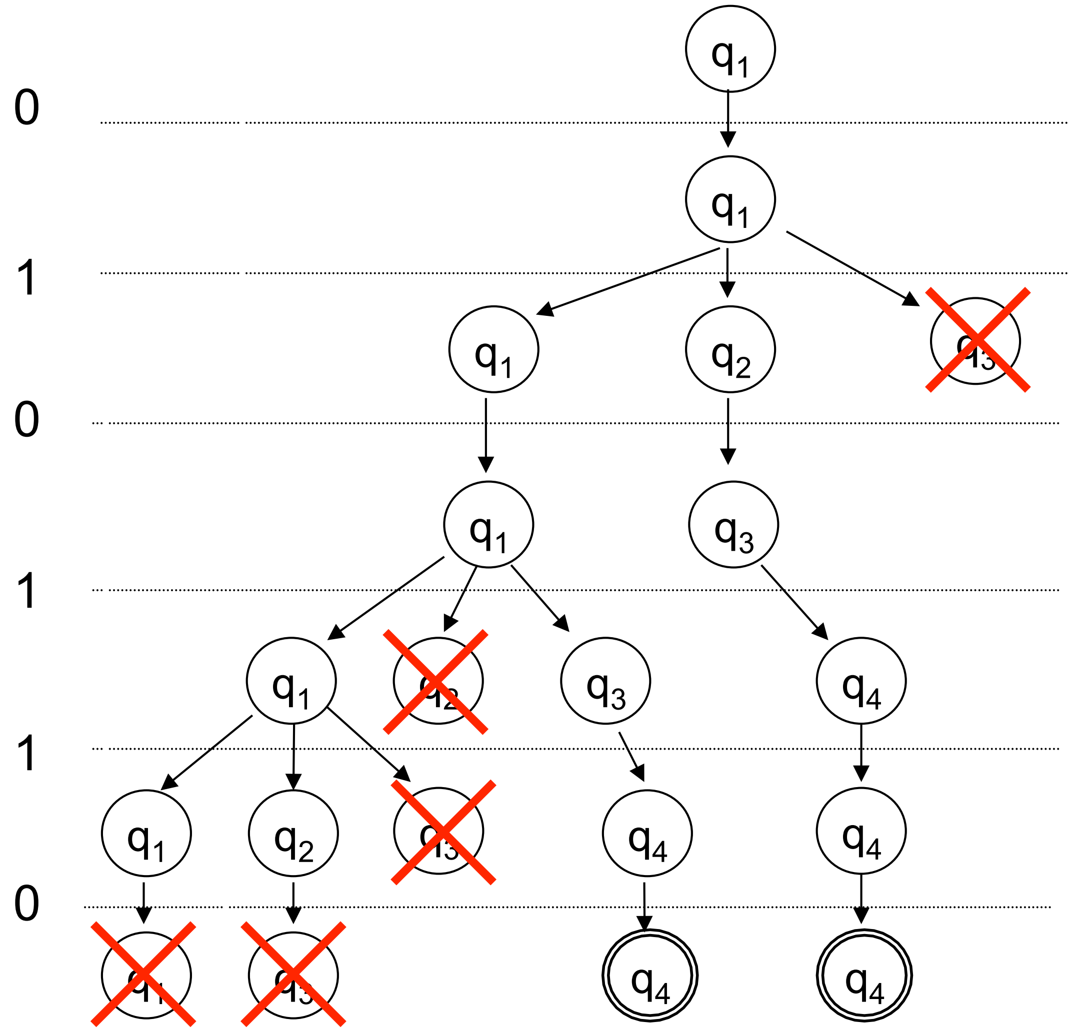


# Example run

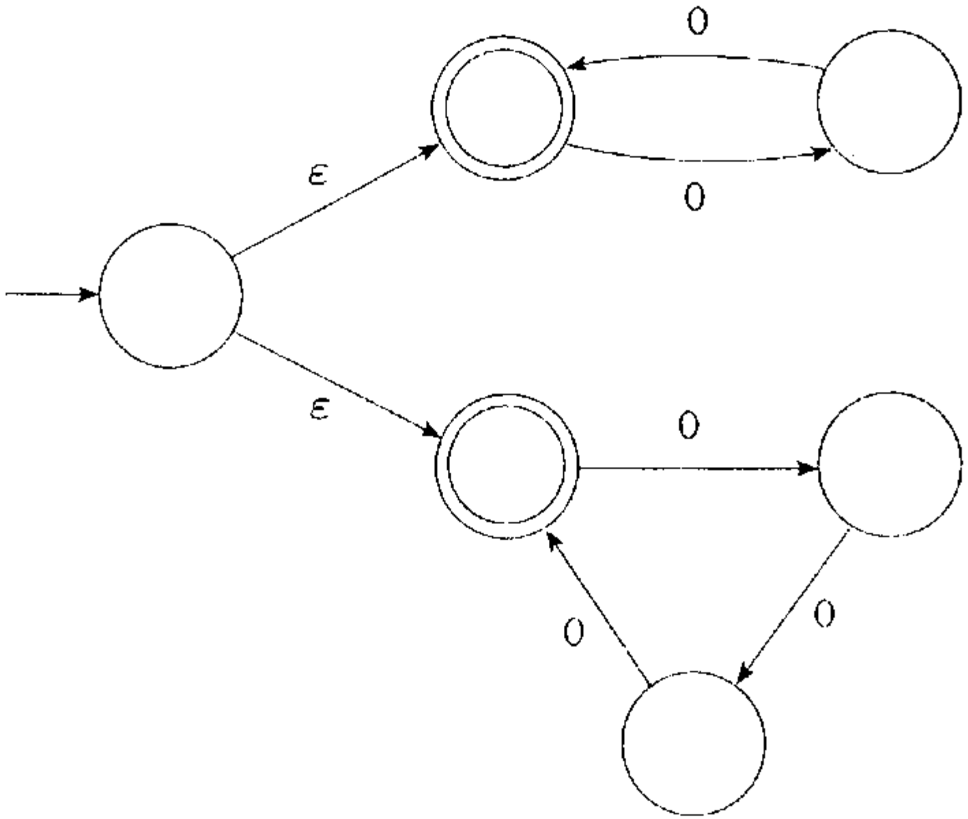



NFA $N_1$

Input: w = 010110



# Another NFA



# Nondeterministic finite automaton

<u>DEFINITION 1.37:</u>

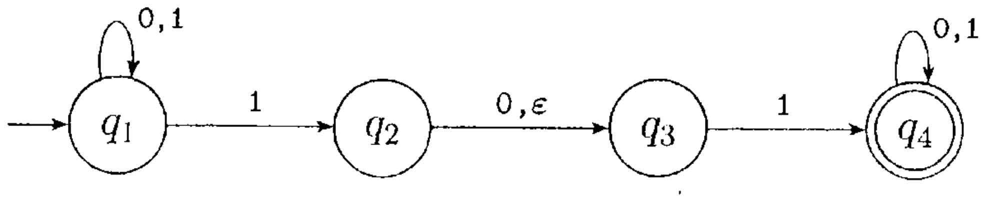
A nondeterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ with:

1. $Q$ a finite set of states
2. $\Sigma$ a finite set, the alphabet
3. $\delta: Q \times \Sigma_\varepsilon \to P(Q)$ is the transition function
4. $q_0 \in Q$ is the start state
5. $F \subseteq Q$ is the set of accept states

$\Sigma_\varepsilon$ includes $\varepsilon$

$P(Q)$ the powerset of $Q$

# Example 1.18




NFA $N_1$

The formal description of $N_1$ is $(Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0, 1\}$
3. $\delta$ is given as

| | 0 | 1 | ε |
|---|---|---|---|
| $q_1$ | $\{q_1\}$ | $\{q_1, q_2\}$ | $\{\}$ |
| $q_2$ | $\{q_3\}$ | $\{\}$ | $\{q_3\}$ |
| $q_3$ | $\{\}$ | $\{q_4\}$ | $\{\}$ |
| $q_4$ | $\{q_4\}$ | $\{q_4\}$ | $\{\}$ |

4. $q_1$ is the start state

# Formal definition of computation

Let $M$ be a finite automaton $(Q, \Sigma, \delta, q_0, F)$.
Let $w = w_1 \dots w_n$ be a string over $\Sigma$.

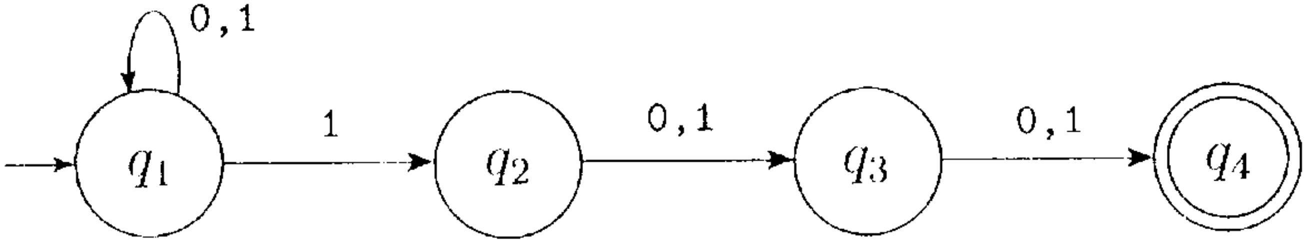
$M$ **accepts** $w$ if a sequence of states $r_0, \dots, r_n$ exists in $Q$ such that

1. $r_0 = q_0$
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for all $i = 0, \dots, n-1$
3. $r_n \in F$

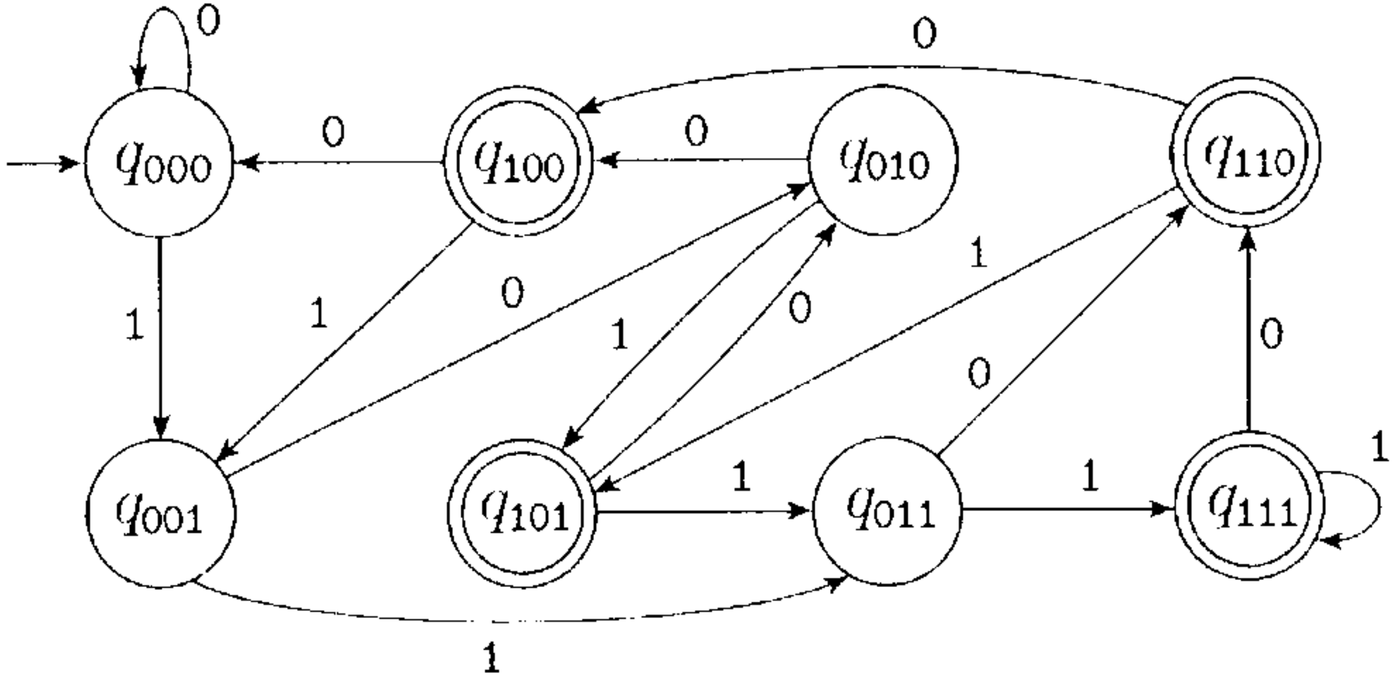
$M$ **recognizes** language $A$ if $A = \{w \mid M \; accepts \; w\}$.

A language is **regular** if some finite automaton recognizes it.

# Every NFA has an equivalent DFA




NFA recognizing language $A$ (figure 1.31)




DFA recognizing language $A$ (figure 1.32)

# Equivalence NFA and DFA

Two machines are **equivalent** if they recognize the same language

Theorem 1.39:

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Corollary 1.40:

A language is regular if and only if some nondeterministic finite automaton recognizes it.

# **Proof:** Theorem 1.39

Let $N = (Q, \Sigma, \delta_0, q_0, F)$ be the NFA recognizing some language A.

Idea: We show how to construct a DFA M recognizing $A$ for any such NFA.

We start by only considering the easier case first, wherein $N$ has no $\varepsilon$ transitions. The $\varepsilon$ transitions are taken into account later.

# **Proof:** Theorem 1.39 (ctd.)

Construct $M = (Q', \Sigma, \delta'_0, q'_0, F')$.

1. $Q' = P(Q)$.
   Every state of $M$ is a set of states of $N$.
   (Recall that $P(Q)$ is the power set of $Q$).
2. For $R \in Q'$ and $a \in \Sigma$ let
   $\delta'(R, a) = \{q \in Q \mid q \in \delta(r, a) \text{ for some } r \in R\}$.
   If $R$ is a state of $M$, it is also a set of states of $N$. When $M$ reads a symbol $a$ in state $R$, it tells us where $a$ takes each state in $R$. Because each state leads to a set of states, we take the union of all these sets. Alternatively we can write:
   $$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$
3. $q'_0 = \{q_0\}$. $M$ starts in the state corresponding to the collection containing just the start state of $N$.

# **Proof:** Theorem 1.39 (ctd.)

4. $F' = \{R \in Q' \mid R\ contains\ an\ accept\ state\ of\ N\}$.
   The machine $M$ accepts if one of the possible states that $N$ could be in at any given moment in an accept state.

The $\varepsilon$ transitions need some extra notation:

a) For any state $R$ of $M$ we define $E(R)$ to be the collection of states that can be reached from $R$ by means of any number of $\varepsilon$ transitions alone, including the members of $R$ themselves. Formally, for $R \subseteq Q$ let

$$E(R) = \{q \mid q\ can\ be\ reached\ from\ R\ along\ 0\ or\ more\ \varepsilon\ transitions\}.$$

b) The transition function $M$ is then modified to take into account all states that can be reached by going along $\varepsilon$ transitions after every step. Replacing $\delta(r, a)$ by $E(\delta(r, a))$ achieves this. Thus,

$$\delta'(R, a) = \{q \in Q \mid q \in E\big(\delta(r, a)\big)\ for\ some\ r \in R\}.$$

# **Proof:** Theorem 1.39 (ctd.)

c) Finally, the start state of $M$ has to cater for all possible states that can be reached from the start state of $N$ along the $\varepsilon$ transitions. Changing $q_0$ to be $E(\{q_0\})$ achieves this effect.

We have now completed the construction of the DFA $M$ that simulates the NFA $N$.

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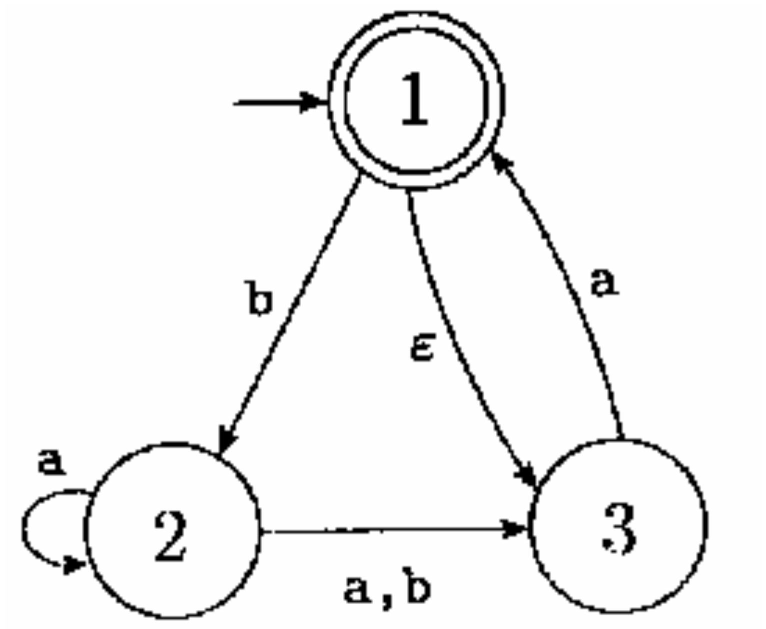
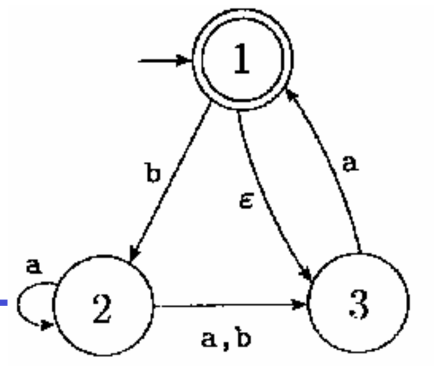



Example:

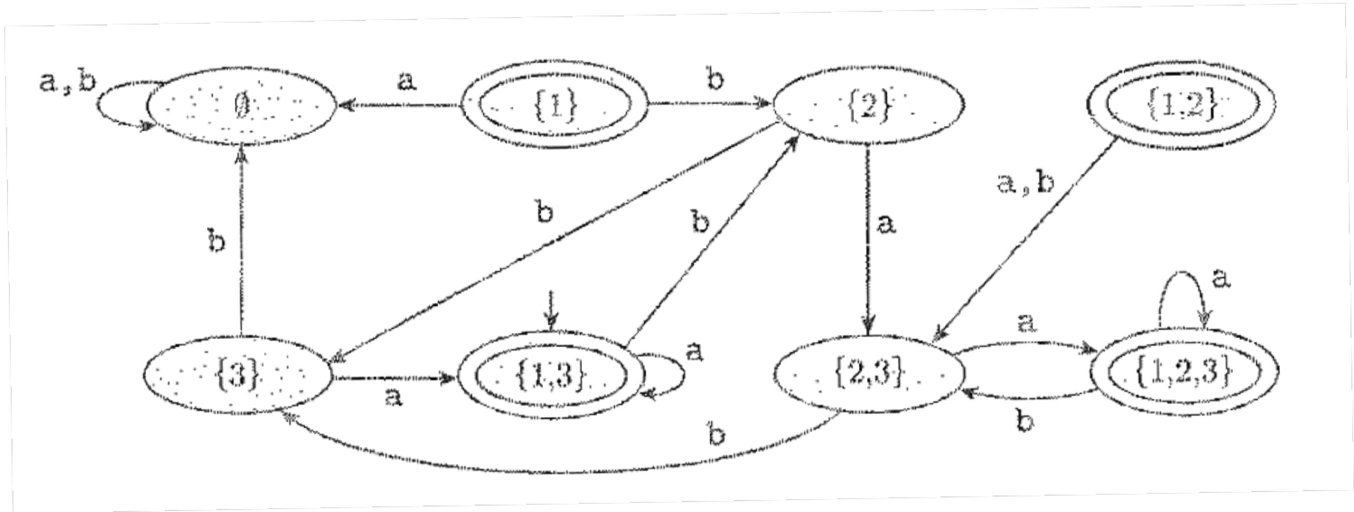
The NFA $N_4$ (figure 1.42)

Construct an equivalent DFA!

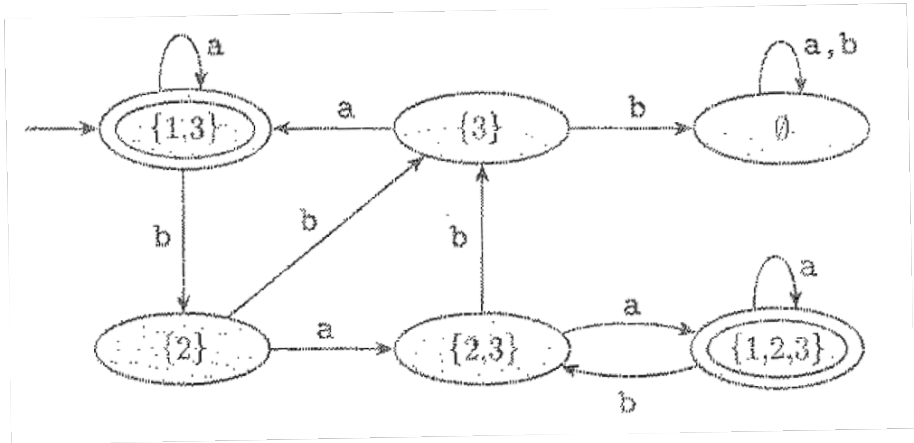
# An example




The resulting DFA




The resulting DFA (after removing redundant states)



# Closure under the regular operations

Theorem 1.45:

The class of regular languages is closed under the union operation. In other words, if $A_1$ and $A_2$ are regular languages, so is $A_1 \cup A_2$.

Theorem 1.47:

The class of regular languages is closed under the concatenation operation.

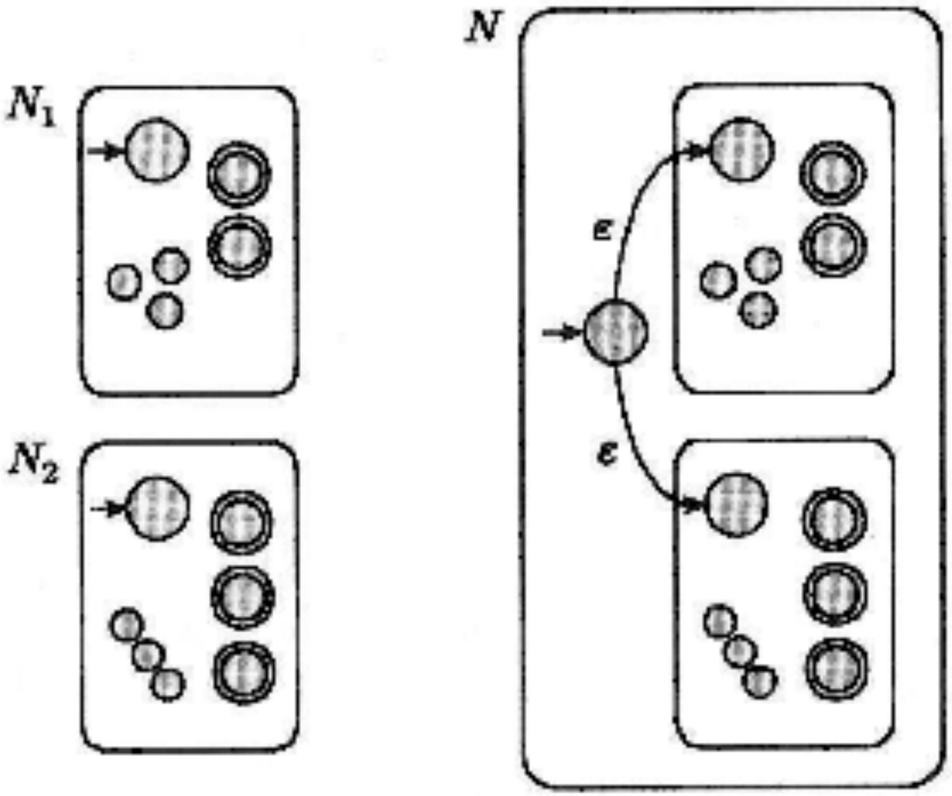
Theorem 1.49:

The class of regular languages is closed under the star operation.

# Proof of Theorem 1.45

The class of regular languages is closed under the union operation.

Idea:



# Proof of Theorem 1.45 (ctd.)

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$ as follows:

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$. The **states** of $N$ are all the states of $N_1$ and $N_2$, with the addition of the new start state $q_0$.
2. The state $q_0$ is the **start state** of $N$.
3. The **accept states** $F = F_1 \cup F_2$. The accept states are all the accept states of $N_1$ and $N_2$. That way $N$ accepts if either $N_1$ or $N_2$ accepts.
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

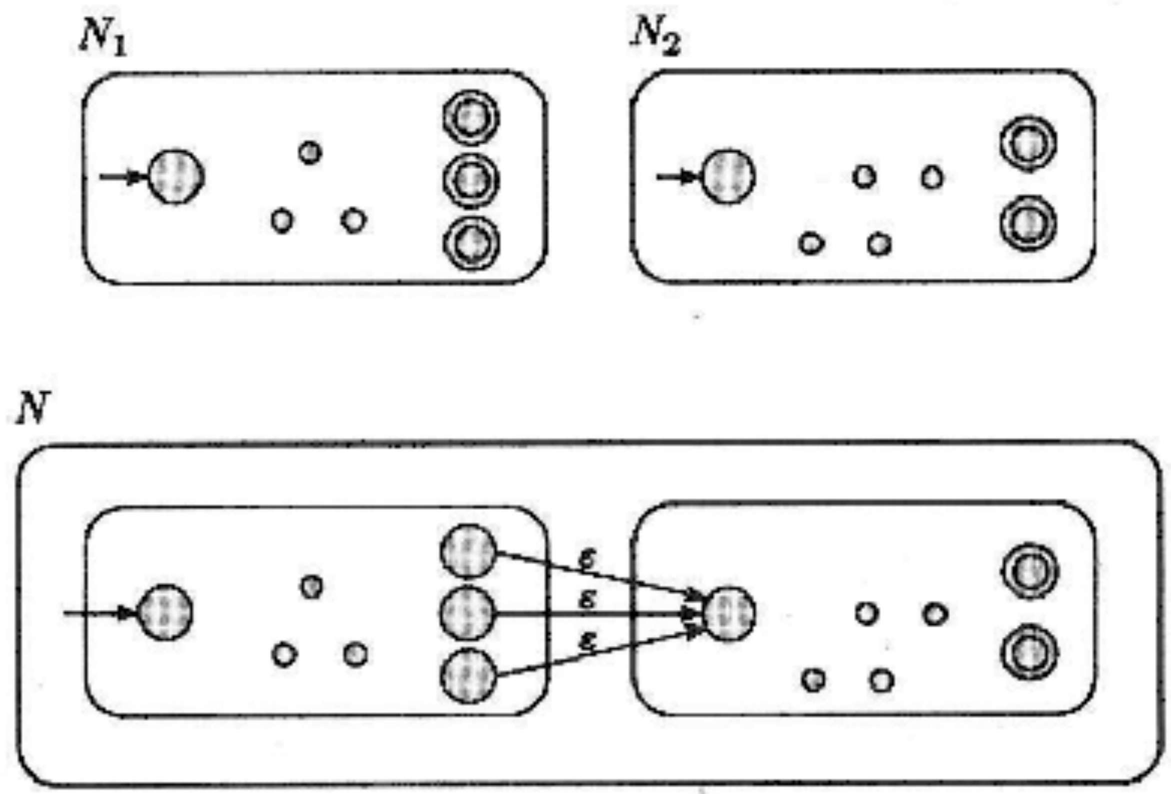
$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

∎

# Proof of Theorem 1.47

The class of regular languages is closed under the concatenation operation.

Idea:



# Proof of Theorem 1.47 (ctd.)

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize $A_2$.

Construct $N = (Q, \Sigma, \delta, \boldsymbol{q_1}, \boldsymbol{F_2})$ to recognize $A_1 \circ A_2$ as follows:

1. $Q = Q_1 \cup Q_2$. The **states** of $N$ are all the states of $N_1$ and $N_2$.
2. The state $q_1$ is the **start state** of $N$, which is the same as the start state of $N_1$.
3. The **accept states** $F_2$ are the same as the accept states of $N_2$.
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

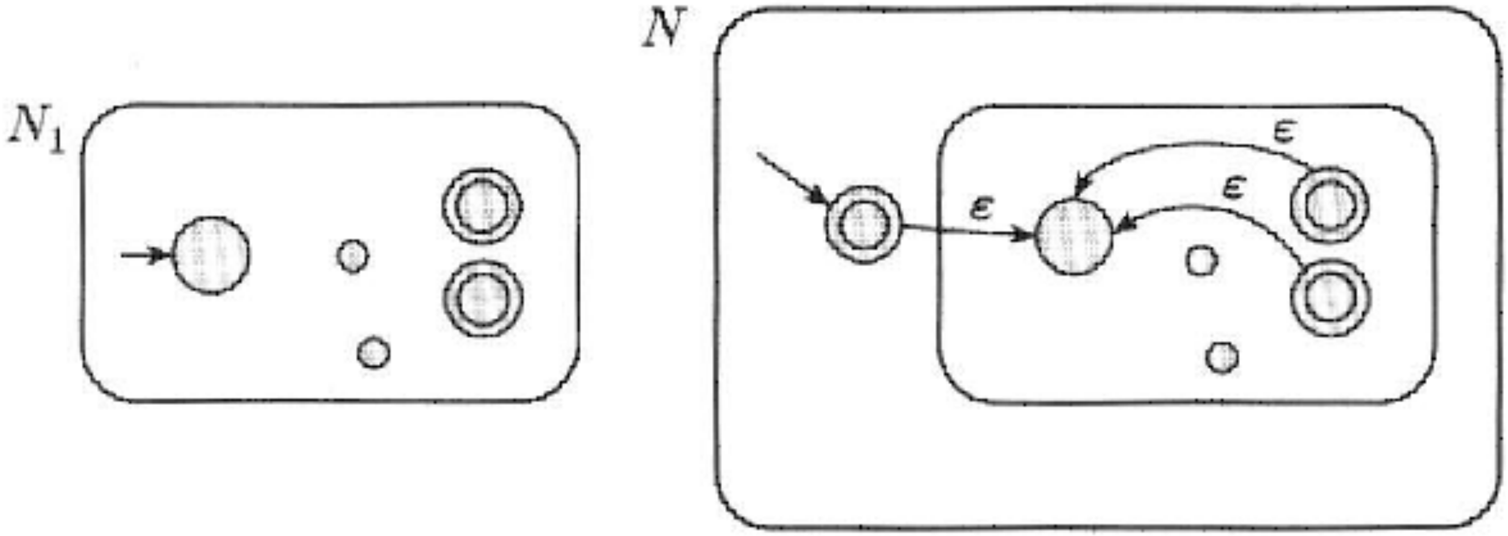
$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \; and \; q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \; and \; a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \; and \; a = \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

■

# Proof of Theorem 1.49

The class of regular languages is closed under the star operation.

Idea:



# Proof of Theorem 1.49 (ctd.)

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize $A_1$.

Construct $N = (Q, \Sigma, \delta, \boldsymbol{q_0}, \boldsymbol{F})$ to recognize $A_1^*$ as follows:

1. $Q = \{q_0\} \cup Q_1$.
   The **states** of $N$ are the states of $N_1$ plus a new start state $q_0$.
2. The state $\mathrm{q_0}$ is the new **start state** of N.
3. $F = \{q_0\} \cup F_1$. The **accept states** are the old accept states plus the new start state.
4. Define $\delta$ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \; and \; q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \; and \; a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \; and \; a = \varepsilon \\ \{q_1\} & q = q_0 \; and \; a = \varepsilon \\ \emptyset & q = q_0 \; and \; a \neq \varepsilon \end{cases}$$

∎

# Regular expressions

DEFINITION 1.52:

Say that $R$ is a **regular expression** if $R$ is

1. $a$ for some $a$ in the alphabet $\Sigma$,
2. $\varepsilon$,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where $R_1$ and $R_2$ are regular expressions,
5. $(R_1 \circ R_2)$, where $R_1$ and $R_2$ are regular expressions, or
6. $(R_1^*)$, where $R_1$ is a regular expression.

# Regular expressions: examples (1)

Let $\Sigma = \{0,1\}$:

1. $0^*10^* = \{w \mid w\ has\ exactly\ a\ single\ 1\}$.
2. $\Sigma^*1\Sigma^* = \{w \mid w\ has\ at\ least\ one\ 1\}$.
3. $\Sigma^*001\Sigma^* = \{w \mid w\ contains\ 001\ as\ a\ substring\}$.
4. $(01^+)^* = \{w \mid every\ 0\ in\ w\ is\ followed\ by\ at\ least\ one\ 1\}$.
5. $(\Sigma\Sigma)^* = \{w \mid w\ is\ a\ string\ of\ even\ length\}$.
6. $(\Sigma\Sigma\Sigma)^* = \{w \mid the\ length\ of\ w\ is\ a\ multiple\ of\ three\}$.
7. $01 \cup 10 = \{01,10\}$.
8. $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 =$
   $\{w \mid w\ starts\ and\ with\ the\ same\ symbol\ as\ it\ ends\}$.

# Regular expressions: examples (2)

Let $\Sigma = \{0,1\}$:

*9.* $(0 \cup \varepsilon)1^* = 01^* \cup 1^*$.
The expression $0 \cup \varepsilon$ describes the language $\{0, \varepsilon\}$, so the concatenation operation adds either 0 or $\varepsilon$ before every string in $1^*$.

*10.* $(0 \cup \varepsilon)(1 \cup \varepsilon) = \{\varepsilon, 0, 1, 01\}$.

*11.* $1^*\emptyset = \emptyset$.
Concatenating the empty set to any set yields the empty set.

*12.* $\emptyset^* = \{\varepsilon\}$.
The star operation puts together any number of strings from the language to get a string in result. If the language is empty, the star operator can only put 0 strings together, giving only the empty string.

# Applications of regular expressions

- Design of compilers

$$\{+, -, \varepsilon\}(DD^* \cup DD^*.D \cup D^*.DD^*)$$
$$where\ D = \{0, \dots, 9\}$$

- awk, grep, vi, ... in *nix systems (search for strings)
- Programming lanugages (e.g. Perl, Python, C++, Java)
- Bioinformatics
  - So-called motifs (patterns occuring in sequences)

# Equivalence of RE and NFA

Theorem 1.54 (page 66):
A language is regular if and only if some regular expression describes it.

*Two directions to consider:*

Lemma 1.55 (page 67):
If a language is described by some regular expression, then it is regular.

Lemma 1.60 (page 69):
If a language is regular, then it can be described by some regular expression.

# Proof of Lemma 1.55

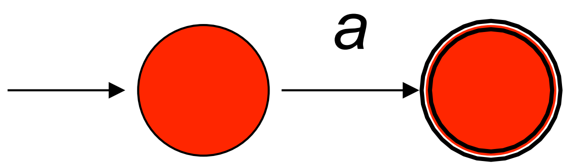
- Idea: Given a regular expression $R$ describing a regular language $A$. We show how to convert $R$ into an NFA recognizing $A$.
- Six cases have to be considered:
  1. $R = a$ for some $a \in \Sigma$, then $L(R) = \{a\}$.
  2. $R = \varepsilon$, then $L(R) = \{\varepsilon\}$.
  3. $R = \emptyset$, then $L(R) = \emptyset$.
  4. $R = R_1 \cup R_2$.
  5. $R = R_1 \circ R_2$.
  6. $R = R_1^*$.

# Proof of Lemma 1.55, case 1

Given: $R = a$ for some $a \in \Sigma$, then $L(R) = \{a\}$

The NFA $N = (\{q_1, q_2\}, \Sigma, \delta, q_1, \{q_2\})$ recognizes $L(R)$ with:

1. $\delta(q_1, a) = \{q_2\}$, and
2. $\delta(r, b) = \emptyset$, for $r \neq q_1$ or $b \neq a$.



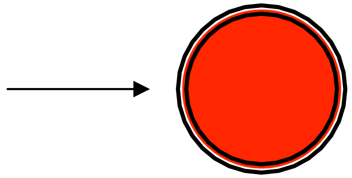

Note: this machine fits the definition of an NFA, but not that of a DFA, as not all input symbols have exiting arrows.

# Proof of Lemma 1.55, cases 2 & 3

Given: $R = \varepsilon$, then $L(R) = \{\varepsilon\}$.

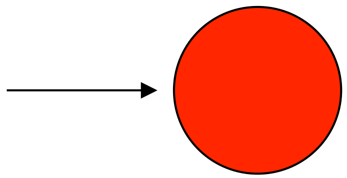
The NFA $N = (\{q_1\}, \Sigma, \delta, q_1, \{q_1\})$
recognizes $L(R)$ with:

1. $\delta(r, b) = \emptyset$, for any $r$ and $b$.



Given: $R = \emptyset$, then $L(R) = \emptyset$.

The NFA $N = (\{q\}, \Sigma, \delta, q, \emptyset)$
recognizes $L(R)$ with:



# Proof of Lemma 1.55, cases 4, 5 & 6

Given:

4. $R = R_1 \cup R_2$.
5. $R = R_1 \circ R_2$.
6. $R = R_1^*$.

The proofs for Theorems 1.45, 1.47, and 1.49 (slide 35, „closure of regular lanugages") can be used to construct the NFA $R$ from the NFAs for $R_1$ and $R_2$ (or just $R_1$ in case 6).

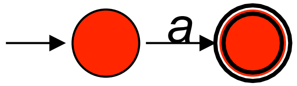
■

# Example 1.56: $(\boldsymbol{ab} \cup \boldsymbol{a})^*$

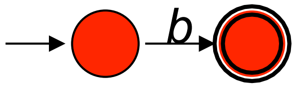
- Convert regular expression $(ab \cup a)^*$ into an NFA in a sequence of stages.
- Build up from the smallest subexpressions to larger subexpressions until NFA for the original expression is achieved.
- Note: This procedure generally does not result in the NFA with the fewest states!

# Example 1.56: NFA for (ab ∪ a)*

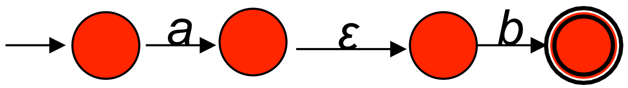
- a:



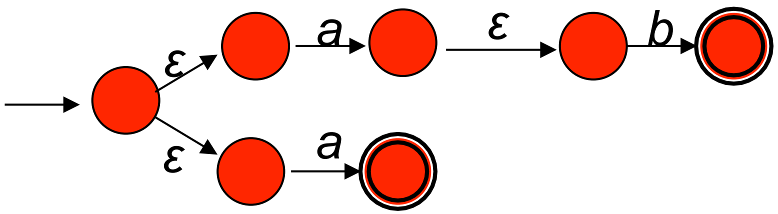

- b:



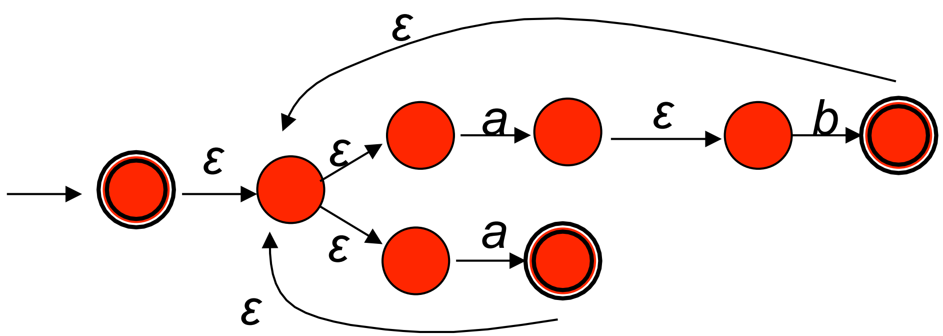

- ab:




- ab ∪ a

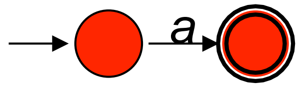



- (ab ∪ a)*

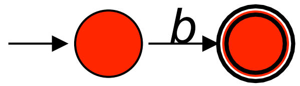


# Exercise: NFA for (a ∪ b)*aba

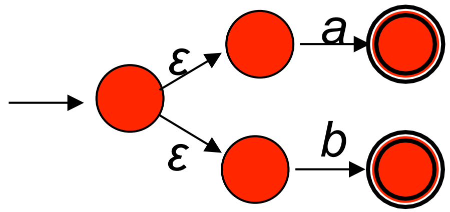
- a:



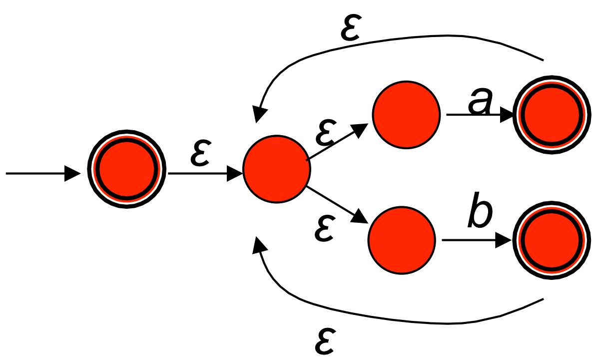

- b:




- a ∪ b

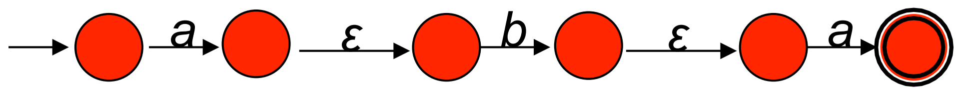



- (a ∪ b)*

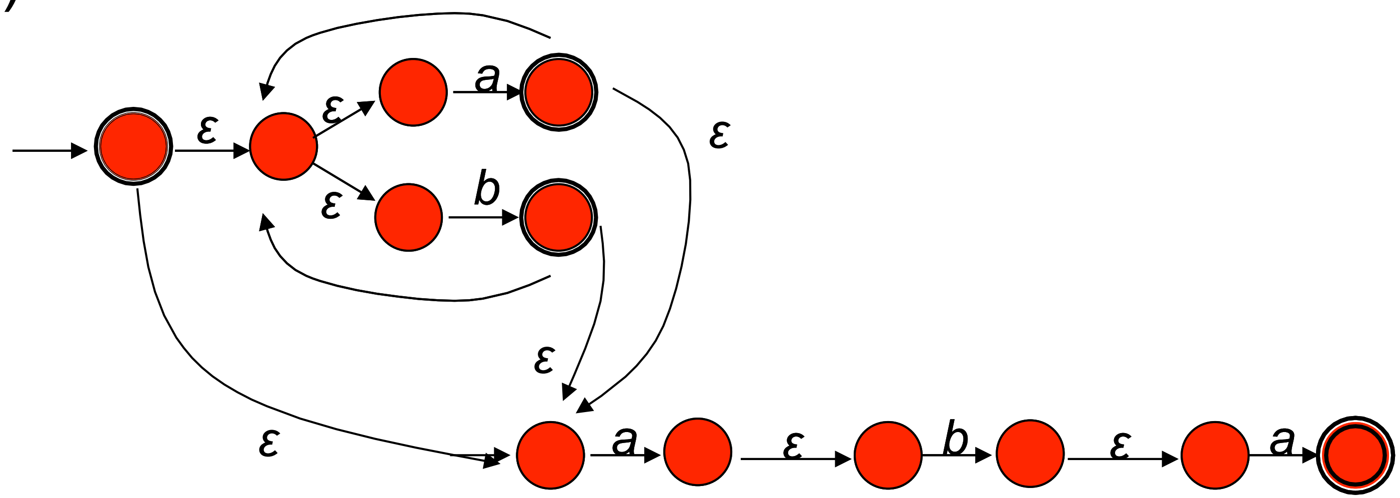


# Example: NFA for (a ∪ b)*aba (cont.)

- aba:




- (a ∪ b)*aba:



# Lemma 1.60

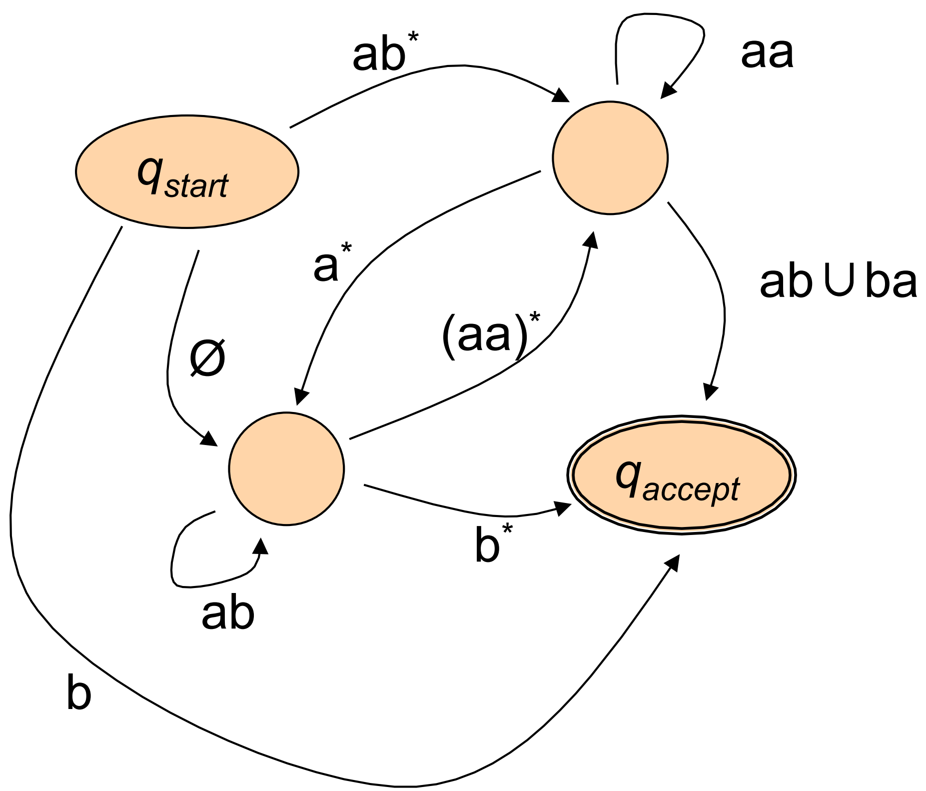
Lemma 1.60 (page 69):

If a language is regular, then it can be described by a regular expression.

- Two steps
  - DFA into GNFA (generalized nondeterministic finite automaton)
  - Convert GNFA into regular expression

# GNFAs

- Labels are regular expressions
- Two states q and r are connected in both directions (fully connected)
- Exception:
  - *One direction only*
  - Start state (exiting transition arrows)
  - Accept state (only one!) (only incoming transition arrows)



# Generalized NFA

A generalized nondeterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where:

1. $Q$ a finite set of states
2. $\Sigma$ a finite set, the alphabet
3. $\delta: (Q \setminus \{q_{accept}\}) \times (Q \setminus \{q_{start}\}) \to \mathcal{R}$ is the transition function
4. $q_{start} \in Q$ is the start state
5. $q_{accept} \in Q$ is the accept state

$\mathcal{R}$ represents the collection of all regular expressions over the alphabet $\Sigma$.

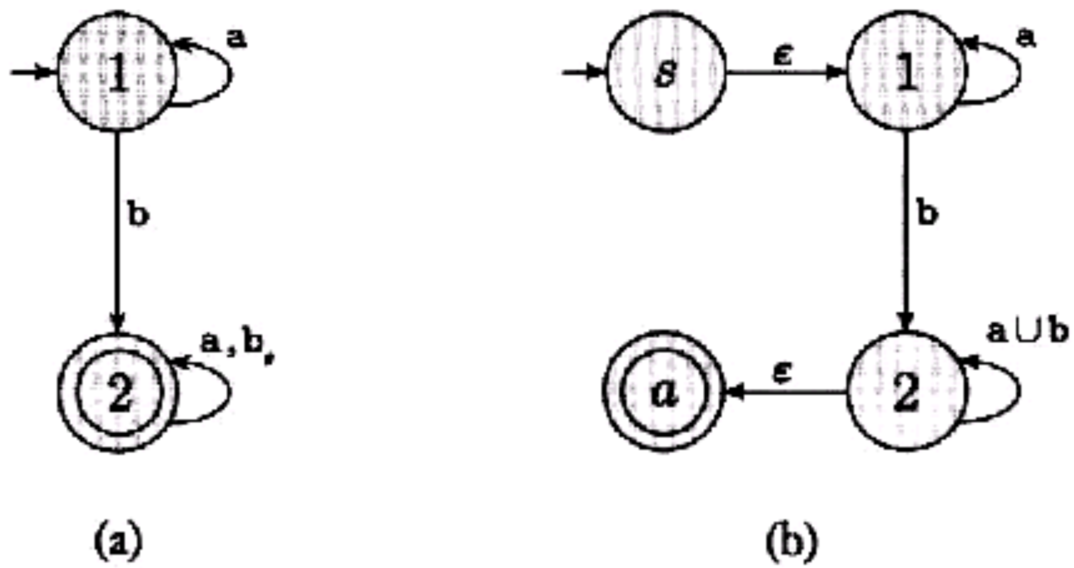
# A generalized NFA accepts string $w$…

A GNFA accepts string $w \in \Sigma^*$ if $w = w_1 w_2 \ldots w_k$, where each $w_i \in \Sigma^*$ and a sequence of states $q_0, q_1, \ldots, q_k$ exists such that

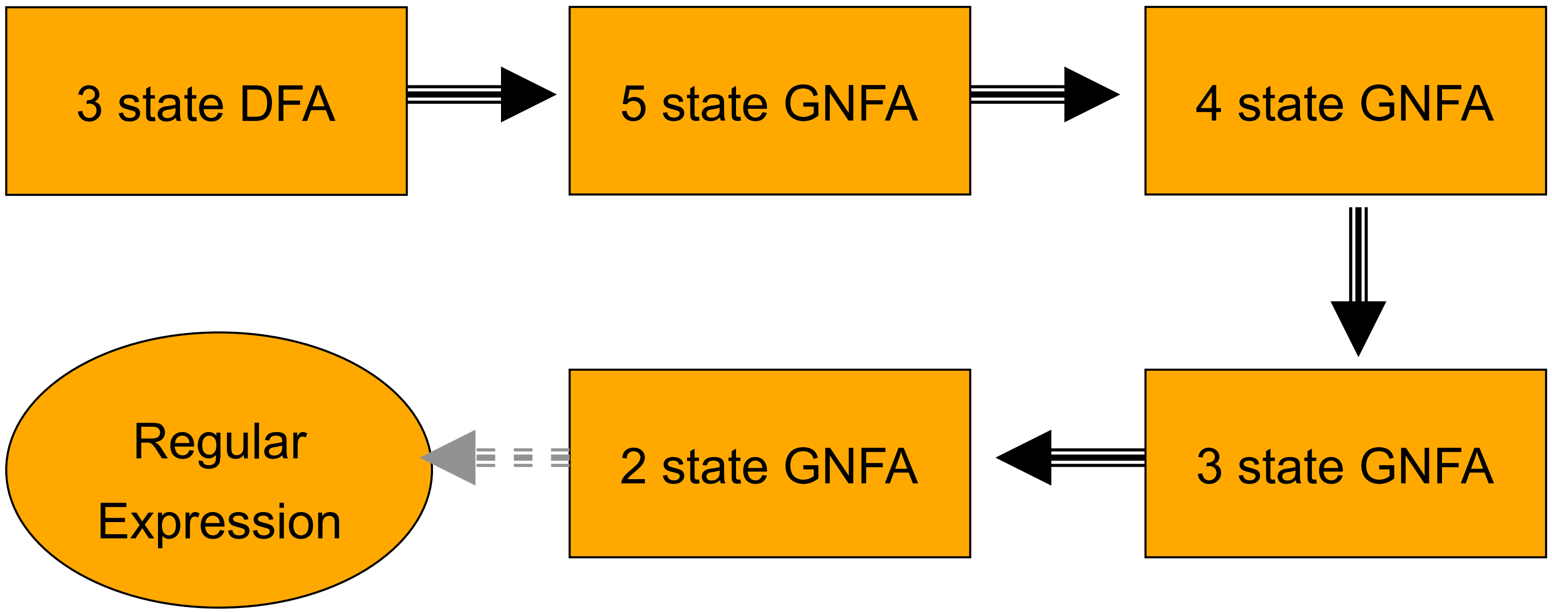
1. $q_0 = q_{start}$ is the start state,
2. $q_k = q_{accept}$ is the accept state, and
3. for each $i$, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$; in other words, $R_i$ is the expression on the arrow from $q_{i-1}$ to $q_i$.

# Convert DFA into GNFA

- add new start state, with $\varepsilon$ transition to old start state
- add new accept state, with $\varepsilon$ transitions from old accept states
- if any transitions have multiple labels $a$ and $b$, replace them by $a \cup b$
- add transitions with label $\emptyset$ between states that had no transitions before

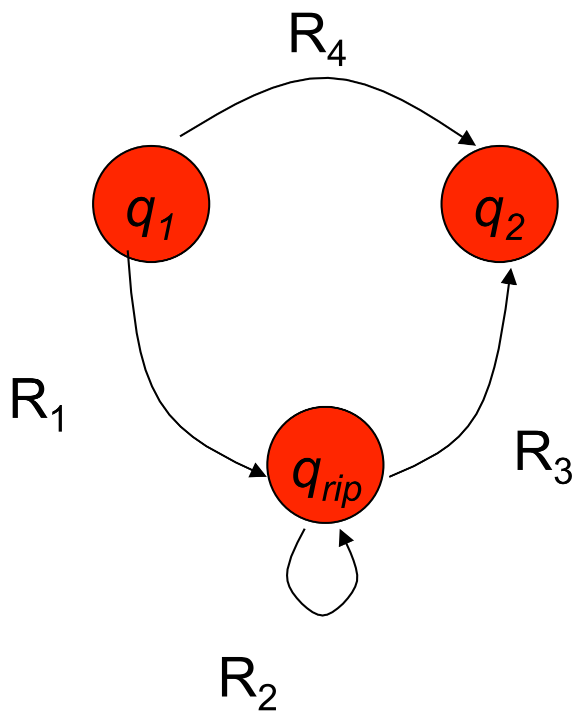


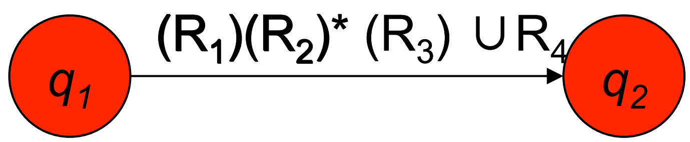
# Convert GNFA into regular expression



# Ripping of states

Replace one state by the corresponding RE



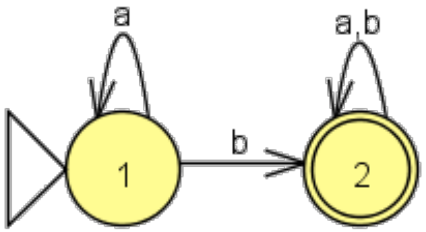



# Convert($G$)

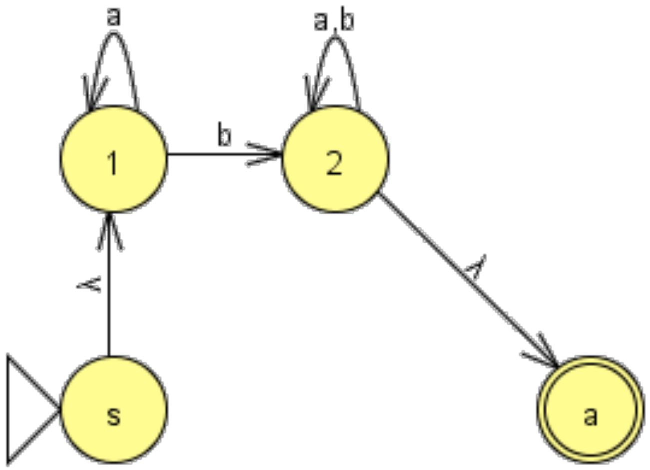
1. Let $k$ be the number of states of $G$.
2. If $k = 2$, then $G$ must consists of a start state, an accept state, and a single transition connecting them, which is labeled with a regular expression $R$. Return the expression $R$ and exit.
3. If $k > 2$, we select any state $q_{rip} \in Q$ different from $q_{start}$ and $q_{accept}$ and let $G'$ ne the GNFA $(Q', \Sigma, \delta', q_{start}, q_{accept})$, where
$$Q' = Q \backslash \{q_{rip}\},$$
and for any $q_i \in Q' \backslash \{q_{accept}\}$ and any $q_j \in Q' \backslash \{q_{start}\}$ let
$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$
for $R_1 = \delta(q_i, q_{rip}), R_2 = \delta(q_{rip}, q_{rip}), R_3 = \delta(q_{rip}, q_j)$, and $R_4 = \delta(q_i, q_j)$.
4. Compute $Convert(G')$ and return this value.

# Example

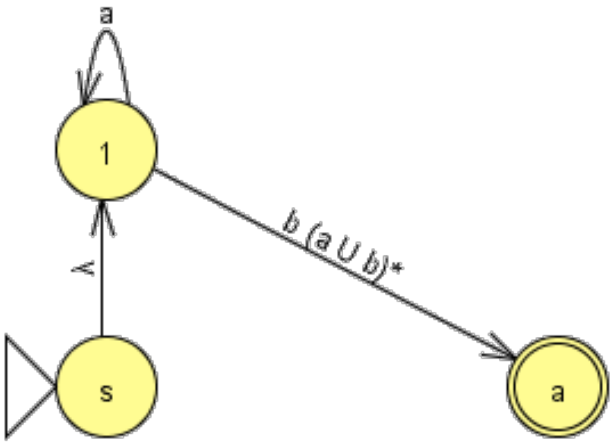
DFA:



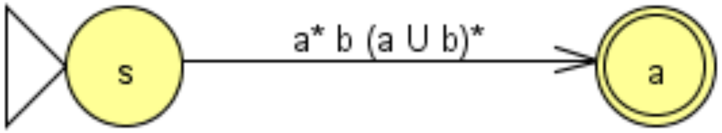

Step 1: convert into GNFA




Step 2: rip state 2

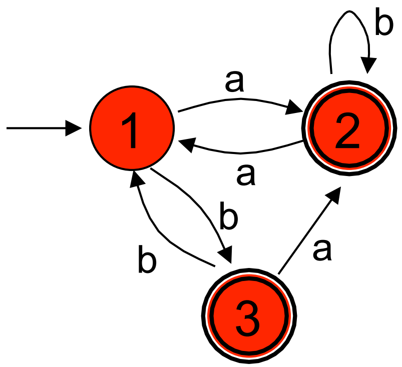



Step 3: rip state 1:

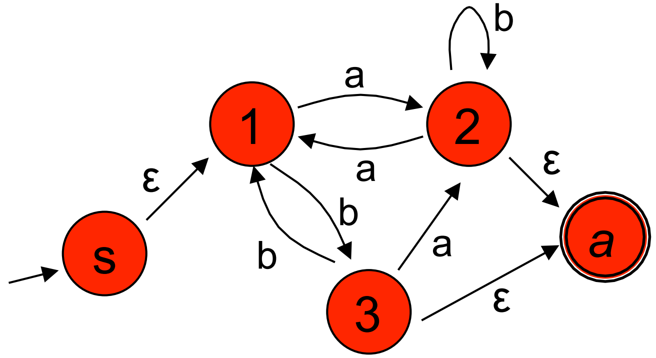


# Another Example

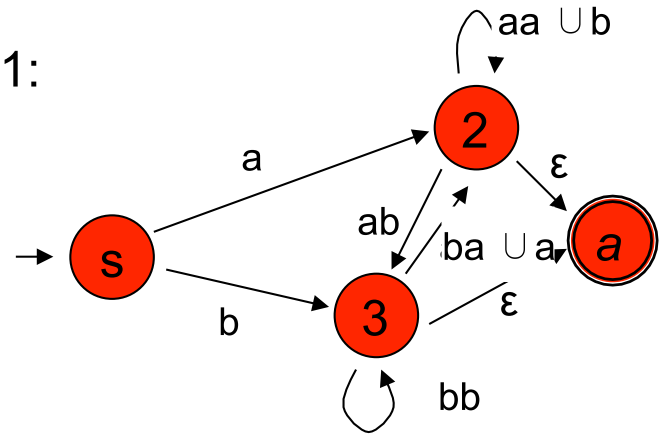
DFA:



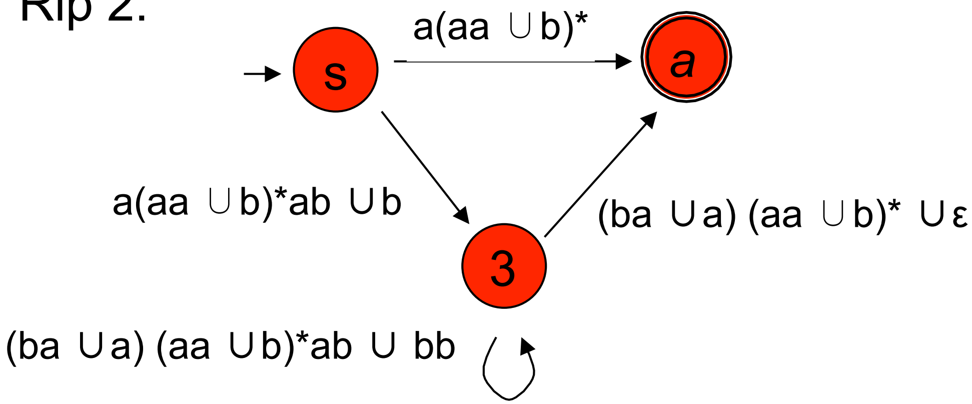

GNFA:



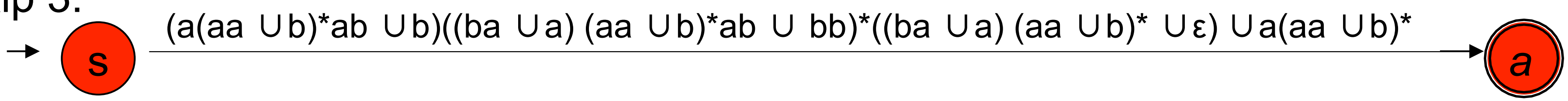

Rip 1:



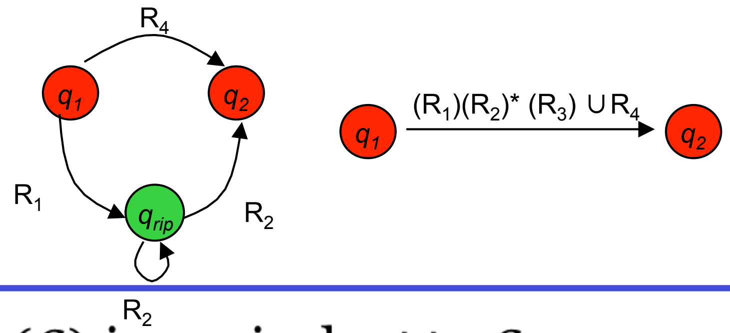

Rip 2:




Rip 3:



# Induction Proof




<u>Claim 1.65:</u> For any GNFA $G$, $Convert(G)$ is equivalent to $G$.

<u>Procedure:</u> We proof this claim by induction on $k$, the number of states of the GNFA.

<u>Basis:</u> Prove the claim true for $k = 2$ states. If $G$ has only two states, it can have only a single transition, which goes from the start state to the accept state. The regular expression label on this transition describes all the strings that allow $G$ to get to the accept state. Hence, this expression is equivalent to $G$.

<u>Induction step:</u> Assume that the claim is true for $k - 1$ states and use this assumption to prove that the claim is true for $k$ states. First we show that $G$ and $G'$ recognize the same language. Suppose that $G$ accepts an input $w$. Then in an accepting branch of the computation $G$ enters a sequence of states

$$q_{start}, q_1, q_2, q_3, \ldots, q_{accept}.$$

# Induction Proof (ctd.)

$$q_{start}, q_1, q_2, q_3, \ldots, q_{accept}.$$

If none of them is the removed state $q_{rip}$, clearly $G'$ also accepts $w$, because each of the new regular expressions labeling the transitions of $G'$ contains the old regular expression as part of a union.

If $q_{rip}$ does appear, removing each run of consecutive $q_{rip}$ states forms an accepting computation for $G'$. The states $q_i$ and $q_j$ bracketing a run have a new regular expression on the transition between them that describes all strings taking $q_i$ to $q_j$ via $q_{rip}$ on $G$. So $G'$ accepts $w$.

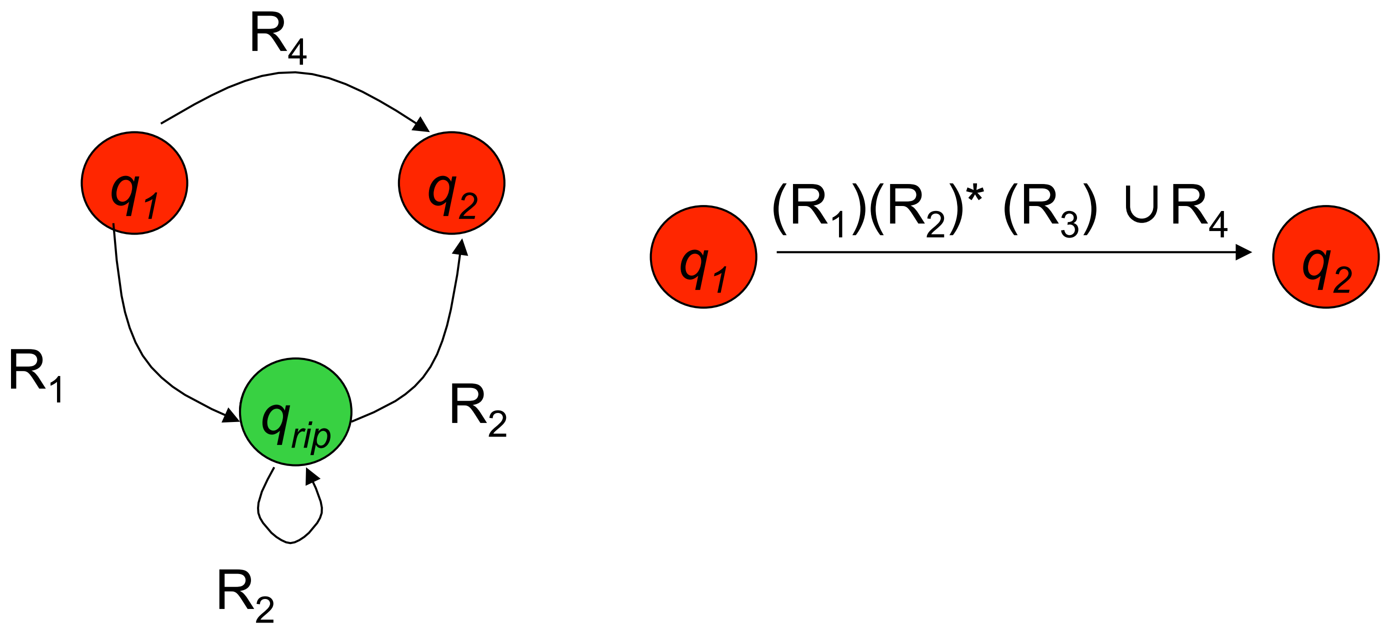
For the other direction, suppose that $G'$ accepts an input $w$. As each transition between any two states $q_i$ and $q_j$ in $G'$ describes the collection of strings taking $q_i$ and $q_j$ in $G$, either directly or via $q_{rip}$, $G$ must also accept $w$. Thus, $G$ and $G'$ are equivalent.

…

# Induction Proof (ctd.)

The induction hypothesis states that when the algorithm calls itself recursively on input $G'$, the result is a regular expression that is equivalent to $G'$, because $G'$ has $k-1$ states. Hence this regular expression also is equivalent to $G$, and the algorithm is proved correct.

■



# Nonregular Languages

- Finite Automata have a finite memory
- Are the following languages regular ?

$B = \{0^n 1^n \mid n \geq 0\}$

$C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$

$D = \{w \mid w \text{ has an equal number of occurences of 01 and 10}\}$

- Mathematical proof necessary

# The pumping lemma

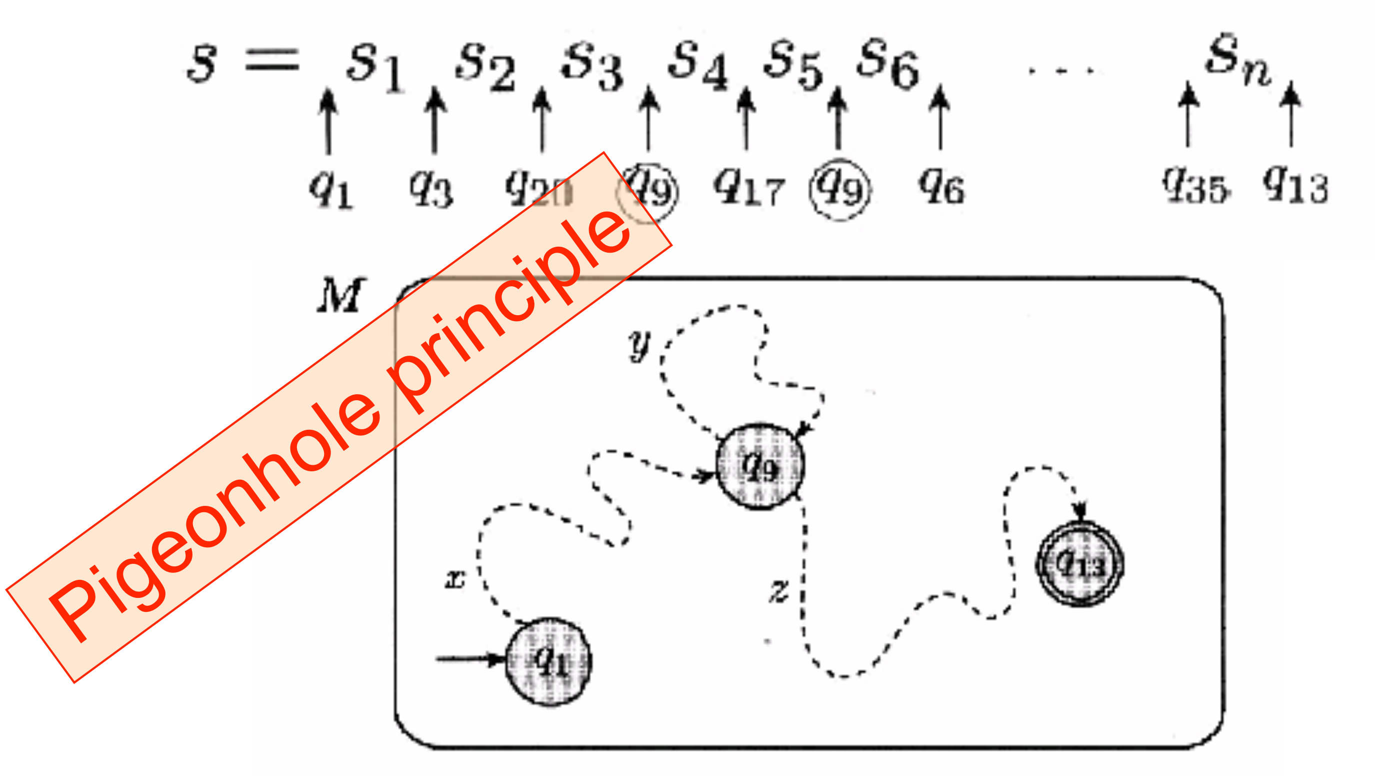
If $A$ is a regular language, then there is a number $p$ (the pumping length), such that any string $s$ of length at least $p$ may be divided into three pieces, $s = xyz$, such that

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Note: from 2 follows that $y \neq \varepsilon$.

# Proof Idea

Let $M$ be a DFA recognizing A.
Assign $\boldsymbol{p}$ **to be the number of states in** $\boldsymbol{M}$**.**
Show that string $\boldsymbol{s}$, with length **at least** $\boldsymbol{p}$, can be broken into $\boldsymbol{xyz}$.




Now prove that all three conditions are met.

# Proof: Pumping Lemma

- Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA recognizing $A$ and $|Q| = p$.
- Let $s = s_1 s_2 \ldots s_n$ be a string in $A$, with $|s| = n$, and $n \geq p$
- Let $r = r_1, \ldots, r_{n+1}$ be the sequence of states that $M$ enters for $s$,

  so $r_{\mathrm{i}+1} = \delta(r_i, s_i)$ with $1 \leq i \leq n$. $|r_1, \ldots, r_{n+1}| = n + 1, n + 1 \geq p + 1$.
- Among the first $p + 1$ elements in r, there must be a $r_j$ and a $r_l$ being the same state $q_m$, with $j \neq l$.

  As $r_l$ occurs in the first $p + 1$ states: $l \leq p + 1$.
- Let $\boldsymbol{x} = s_1 \ldots s_{j-1}$, $\boldsymbol{y} = s_j \ldots s_{l-1}$ and $\boldsymbol{z} = s_l \ldots s_n$:
  - as $\boldsymbol{x}$ takes $M$ from $r_1$ to $r_j$, **y** from $r_j$ to $r_l$, and **z** from $r_l$ to $\mathrm{r}_{\mathrm{n}+1}$, being an accept state, M must accept $xy^iz$ for $i \geq 0$.
  - with $j \neq l, |y| > 0$
  - with $l \leq p + 1, |xy| \leq p$

# Pumping Lemma (cont.)

Use pumping lemma to prove that a language $A$ is not regular:

1. Assume that $A$ is regular (Proof by contradiction)
2. use the lemma to guarantee the existence of $p$, such that strings of length $p$ or greater can be pumped
3. find string $\boldsymbol{s}$ of $A$, with $|s| \geq p$ that cannot be pumped
4. demonstrate that s cannot be pumped using ***all different ways of dividing s into x,y, and z*** (using condition 3 is here very useful)
5. the existence of $s$ contradicts the assumption, therefore $A$ is not a regular language

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

# Nonregular languages: example 1

$$B = \{0^n 1^n \mid n \geq 0\}$$

- Choose string $s = 0^p 1^p$ for $p \in \mathbb{N}^+$ being the pumping length
- If we were to consider condition 2, then we would have that:
  1. string $y$ consists only of 0s → $xyyz$ has more 0s than 1s → not a member of $B$ → violates condition 1 → contradiction!
  2. string $y$ consists only of 1s → similar argument as in case 1 → contradiction!
  3. string $y$ consists of both 0s and 1s → $xyyz$ may have same number of 0s and 1s, but out of order with some 1s before 0s → contradiction!

Intuitive argument: A DFA $M$ would need to be able to remember how many 0s have been seen so far as it reads the input. As the number of 0s isn't limited and all DFAs only have a finite number of states, $B$

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

# Nonregular languages: example 2

$$C = \{w \mid w \text{ has an equal number of } 0s \text{ and } 1s\}$$

- Choose string $s = 0^p1^p$ for $p \in \mathbb{N}^+$ being the pumping length
- Pumping $s$ seems possible, but only if we ignored condition 3!
  - Condition3: $|xy| \leq p$
  - Thus, $y$ consists of 0s only
  - Then $xyyz \notin C$ → Contradiction!

Alternative proof:

- We know that $B = \{0^n1^n \mid n \geq 0\}$ is not regular.
- If $C$ were regular, then $C \cap 0^*1^* = B$ also regular, because regular languages are closed under intersection (cp. slide 14)!
  → Contradiction!

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

# Nonregular languages: example 3

$$F = \{ww \mid w \in \{0,1\}^*\}$$

- Choose string $s = 0^p1^p$ for $p \in \mathbb{N}^+$ being the pumping length
  - Does NOT WORK, because it CAN be pumped! Try again..
- Choose string $s = 0^p10^p1$ for $p \in \mathbb{N}^+$ being the pumping length
- We use condition 3 again:
  - Condition3: $|xy| \leq p$
  - Thus, $y$ consists of 0s only
  - Then $xyyz \notin F$ → Contradiction!
- Choice of $s$ is crucial
  - If some $s$ does not work, try another one!

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

# Nonregular languages: example 4

$$E = \{0^i 1^j \mid i > j\}$$

- Choose string $s = 0^{p+1}1^p$ for $p \in \mathbb{N}^+$ being the pumping length
- We use condition 3 again:
  - Condition3: $|xy| \leq p$
  - Thus, $y$ consists of 0s only
  - Then $xy^0z = xz \notin E$ → Contradiction!
- Here we use $xy^0z$ instead of $xyyz$ as argument. This is commonly called „pumping down“.

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

# Example exam question

Q: Use the pumping lemma to prove that
$L = \{0^k 1^j \mid k, j \geq 0 \text{ and } k \geq 2j\}$ is not regular.

A: Assume that $L = \{0^k 1^j \mid k, j \geq 0 \text{ and } k \geq 2j\}$ is regular. Let $p$ be the pumping length of $L$. The pumping lemma states that for any string $s \in L$ of at least length $p$, there exist strings $x, y$, and $z$ such that $s = xyz$, $|xy| \leq p$, $|y| > 0$, and for all $i \geq 0$: $xy^iz \in L$.

Choose $s = 0^{2p}1^p$. Because $s \in L$ and $|s| = 3p \geq p$, we obtain from the pumping lemma the strings $x, y$, and $z$ with the above properties. As $s = xyz$, $|xy| \leq p$, and $s$ begins with $2p$ zeros, one can see that $xy$ can only consist of zeros. If we pump $s$ down, i.e. select $i = 0$, the string $xy^0z = xz = 0^{2p-|y|}1^p$.

As $xz$ has $p$ ones, and $|y| > 0$, $xz$ has fewer than $2p$ zeros.

Hence $xz \notin L \Rightarrow$ CONTRADICTION.

Therefore $L$ is not regular!

# Summary

- Deterministic finite automata
- Regular languages
- Nondeterministic finite automata
- Closure operations
- Regular expressions
- Nonregular languages
- The pumping lemma