Motivation

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Course Content

- *Introduction to logic*
  - *Propositional*
  - *First order logic*

- *Theoretical foundations of computer science*
  - *Automata Theory*
  - *Formal languages, grammars*
  - *Decidability*
  - *Computational Complexity*

Theoretical Computer Science motivation

- *Overall question :*
  - *What are the fundamental capabilities and limitations of computers ?*
- *Subquestions :*
  - *What is the meaning of computation ?*
    - *Automata theory*
  - *What can be computed ?*
    - *Computability/Decidability theory*
  - *What can be computed efficiently ?*
    - *Computational complexity*

What is the meaning of computation ?

- *1930-50s : Automata theory ?*
  - *Various mathematical models of computers*
    - *Automata theory*
    - *Turing Machines*
    - *Grammars (Noam Chomsky)*
    - *Practical :*
      - *Many devices (dishwashers, telephones, …)*
      - *Compilers and languages*
      - *Protocols*

What can be computed ?

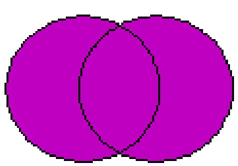
- *What can be computed using Turing Machines?*
  - *Some problems can be solved algorithmically*
    - *E.g. sorting*
  - *Others cannot :*
    - *E.g. the halting problem determine whether Turing machine M accepts w or not*
      - *May not terminate (if M loops)*
    - *E.g. Goedel : no algorithm can decide in general whether statements in number theory are true or false*
  - *Practical :*
    - *It is important to know what can be computed and what not*

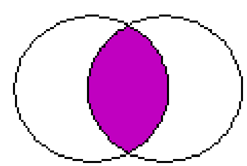
What can be computed efficiently ?

- *Examples*
  - *Sorting can be done efficiently*
  - *Scheduling cannot be done efficiently*
    - *University lectures*
  - *Complexity theory gives an explanation*
    - *NP-hard problems*
  - *Practical :*
    - *Important to know how hard your problem is*
    - *Cryptography*

Some mathematical concepts: Sets

- A set is a group of objects
  - $\{4,7,12\}$, the empty set is denoted $\emptyset$ or $\{\}$
- Membership is denoted with $\in$ and $\notin$:
  - $4 \in \{4,7,12\}$ and $5 \notin \{4,7,12\}$
- Subset $\subseteq$ and propper subset $\subset$:
  - $\{12,4,7\} \subseteq \{4,7,12\}$ and $\{4,7\} \subset \{4,7,12\}$
- Union ($\cup$) and intersection ($\cap$):
  - $A \cup B$ and $A \cap B$





Mathematical concepts: Sequences and Sets

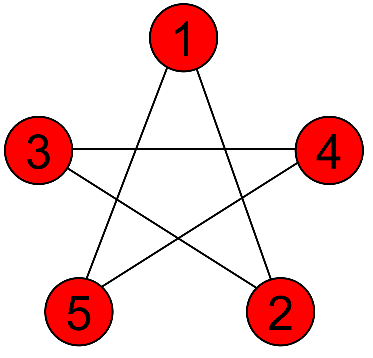
- Sequence is a list of object in some order:
  - $(4,7,12)$ is not the same as $(12,7,4)$
- Finite or infinite sequences:
  - finite are often called *tuples*, or *k-tuples* (a tuple with *k* elements). A 2-tuple is called a *pair.*
- Power set
  - $A = \{0,1\}$ the power set $A^P = \{\{\},\{0\},\{1\},\{0,1\}\}$
- Cartesian product or cross product
  - $A = \{a,b\} \;\; and\; B = \{1,2,3\}$

    $A \; x \; B = \{\{a,1\},\{a,2\},\{a,3\},\{b,1\},\{b,2\},\{b,3\}\}$

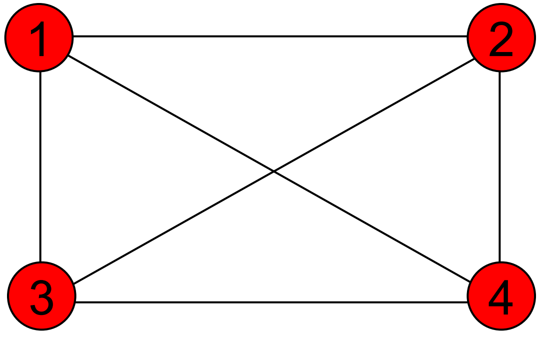
Some mathematical concepts: Graphs

- Graphs $G = (V, E)$

$$G_1 = (\{1,2,3,4,5\}, \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\})$$

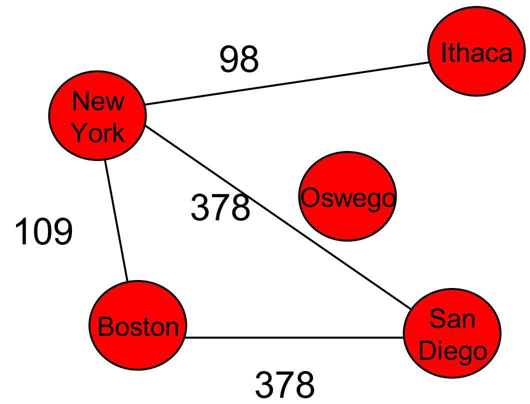


$$G_2 = (\{1,2,3,4\}, \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\})$$

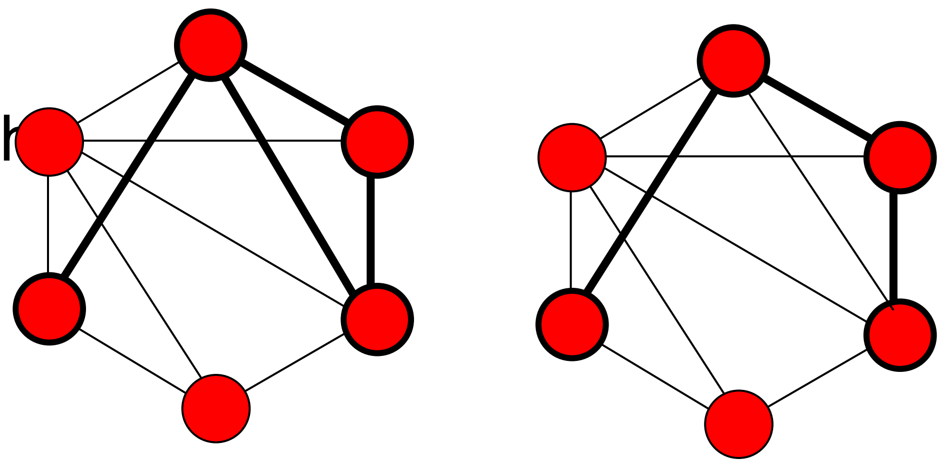


Some mathematical concepts: Graphs II

- Labelled

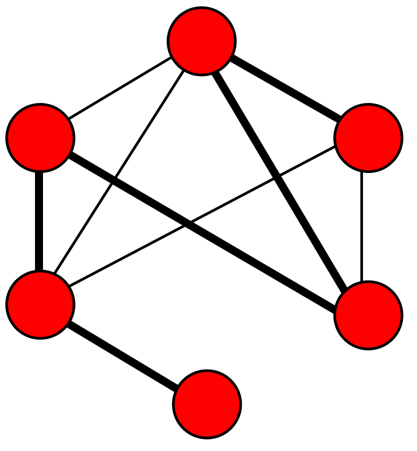


- Subgraph
  induced subgraph

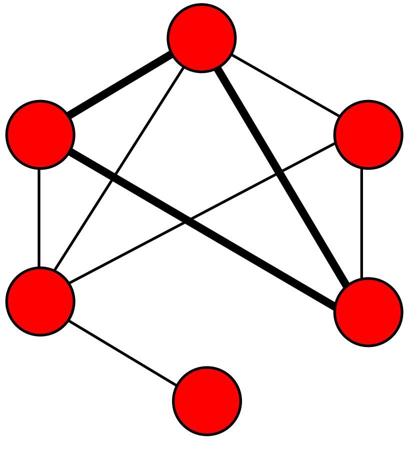


Some mathematical concepts: Graphs III

- Path

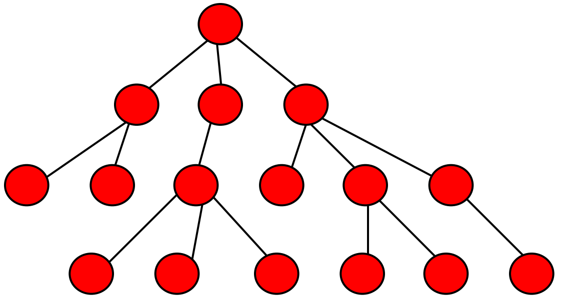


- Cycle

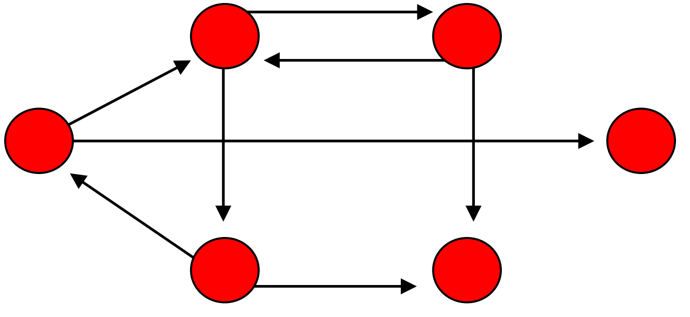


Some mathematical concepts: Graphs IV

- Trees



- Directed Graph



Strings and Languages

- Alphabet = set of symbols
  - e.g.: $\Sigma = \{a, b, c\}$
- String = sequence of symbols over alphabet
  - e.g. aabbabcca
- Length $|w|$ = number of symbols in w
- Empty string = ε
- *aabb* is substring of *aaabbbbccc*
- *xy* concatenation of two strings *x* and *y*
- $x^k = x \ldots x$ (z.B. $x^3 = xxx$)
- Language is a set of strings (over an alphabet Σ)

Mathematical proofs

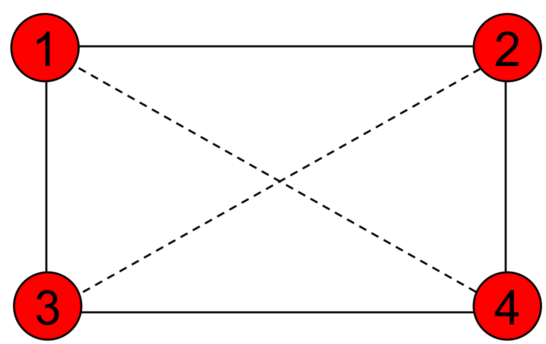
- *Various types of proofs*
  - *Direct proof*
  - *Proof by construction/counterexample*
  - *Proof by contradiction (indirect proof, reductio ad absurdum)*
  - *Proof by induction*
- *How formal?*
  - *Formal enough to be convincing to your audience*

Direct proof

- Strategy: Logically derive conclusions from your premises until you arrive at the desired conclusion.
- Example:
  Let a, b, c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.
- Proof:
  - From $a \mid b$, we get: (1) ex. integer k_1 s.t. $b = k_1 \cdot a$
  - From $b \mid c$, we get: (2) ex. integer k_2 s.t. $c = k_2 \cdot b$
  - From (1) and (2) we get: (3) ex. integers k_1, k_2 s.t. $c = k_2 \cdot k_1 \cdot a$
  - From (3) we get: (4) ex. integer k s.t. $c = k \cdot a$ (namely, $k = k_2 \cdot k_1$)
  - From (4) we get that $a \mid c$.

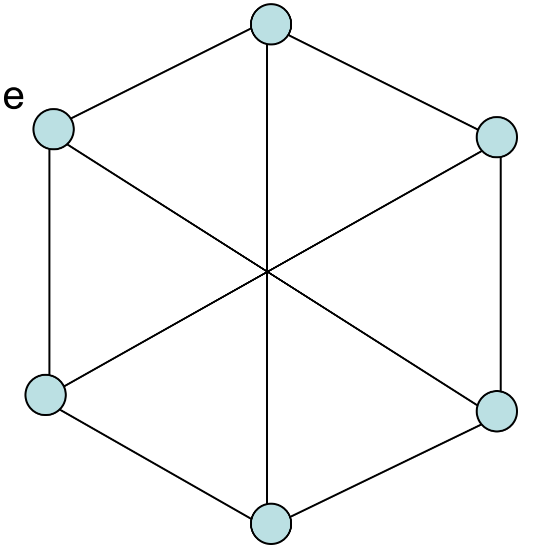
Proof by construction

- *Objective: prove that a particular type of object exists*
  - *Proof strategy: Demonstrate how to construct the object.*
- Example:
  - Definition: A graph is k-regular if all vertices have degree k
  - *Theorem:* For all even numbers $n > 2$, there exists a 3-regular graph with n nodes



Proof by Construction II

- Proof: (Let n > 2 be an even number.)
  - $G = (V, E)$ with
    - $V = \{0,1,\ldots,n-1\}$ and
    - $E = \{\{i, i+1\} \mid for\ 0 \leq i \leq n-2\} \cup \{\{n-1, 0\}\} \cup$
      $\{\{i, i+n/2\} \mid for\ 0 \leq i \leq n/2-1\}\}$
    - → every vertex has exactly three neighbours:
      - its predecessor in the cycle $0, 1, 2, .., n-1, 0$
      - its successor in the cycle
      - its „mirror image“ $n/2$ positions before/ahead in the cycle



- Why do we need $n > 2$ as a requirement?

Proof by contradiction

- **Theorem**: $\sqrt{2}$ is irrational
- **Proof strategy:**
  - Assume that the theorem is not true.
  - Show that this leads to a contradiction, and hence the theorem must be true.

Proof by Contradiction

- **Theorem**: $\sqrt{2}$ is irrational
- **Proof**: Assume the theorem is not true, then:

$\sqrt{2} = \frac{b}{a}$ where a and b are integers and $\frac{b}{a}$ is reduced.

$2 = \frac{b^2}{a^2}$

$2a^2 = b^2$ hence, b^2 is even, hence b is even

now, we can write $b = 2c$, *which gives:*

$2a^2 = 4c^2$ divide by 2, gives:

$a^2 = 2c^2$ hence, a^2 is even, hence a must be even

CONTRADICTION

Proof by induction

- *Prove a statement $S(X)$ about a family of objects (e.g. integers, trees) in two parts :*
  - *Basis: prove for one or several small values of X directly*
  - *Inductive step: Assume $S(Y)$ for Y smaller than X; prove $S(X)$ using that assumption*
- *Applies to*
  - *Natural numbers*
  - *Inductively defined objects (structured induction)*

Inductively defined: example

Rooted binary trees are inductively defined

- **Basis**: a single node is a tree and that node is the root of the tree
- **Induction**: if T_1 and T_2 are rooted binary trees, then the tree constructed as follows is a rooted binary tree:
  - Begin with a new node N
  - Add copies of T_1 and T_2
  - Add edges from N to T_1 and T_2

Proof by induction: example

Theorem: A binary tree with n leaves has $2n-1$ nodes

- Basis:
  - if a tree has one leaf, then it is a one node tree $2*1-1=1$
- Induction:
  - assume $S(T)$ for trees with fewer nodes than T, in particular for subtrees of T *(i.e. use the theorem as an assumption, and use the smaller trees of T, namely U and V to prove it)*
  - T must be a root plus two subtrees U and V
  - If U and V have x and y leaves respectively and T has z leaves, then $z = x + y$
  - By the induction assumption, U and V have $2x-1$ and $2y-1$ nodes, resp.
  - Then T has $1+(2x-1)+(2y-1)$ nodes:

$$1+(2x-1)+(2y-1)$$
$$=2(x+y)-1$$
$$=2z-1$$

$(q.e.d.)$