Constraint Satisfaction Problems

Qualitative Representation and Reasoning

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based on a slideset by

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Quantitative vs. Qualitative Representations

Spatio-temporal configurations can be described quantitatively by specifying the coordinates of the relevant objects:

Example: At time point 10.0 object A is at position $(11.0, 1.0, 23.7)$, at time point 11.0 at position $(15.2, 3.5, 23.7)$. From time point 0.0 to 11.0, object B is at position $(15.2, 3.5, 23.7)$. Object C is at time point 11.0 at position $(300.9, 25.6, 200.0)$ and at time point 35.0 at $(11.0, 1.0, 23.7)$.

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

Example: Object A hit object B. Afterwards, object C arrived.

Sometimes we want to reason with such descriptions.

Example: Object C was not close to object A, when it hit object B.

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Representation of Qualitative Knowledge

Intention: describe configurations in an infinite (continuous) domain using a finite vocabulary and reason about these descriptions

- Specification of a vocabulary: usually a finite set of relations (often binary) that are pairwise disjoint and jointly exhaustive
- Specification of a language: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal semantics
- Analysis of computational properties and design of reasoning methods (often constraint propagation)
- Perhaps, specification of operational semantics for verifying whether a relation holds in a given quantitative configuration

Applications in . . .

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- . . .

Example: Qualitative Temporal Relations

Suppose, we want to talk about time instants (points) and binary relations over them.

- Vocabulary: $X = Y$ (X equals Y), $X < Y$ (X before Y), and $X > Y$ (X after Y).
- Language:
  - Allow for disjunctions of basic relations to express indefinite information. Use unions of relations to express that. For instance, $< \cup =$ expresses $\leq$.
  - 2^3 different relations (including the impossible and the universal relation)
  - Use sets of atomic formulae with these relations to describe configurations. For example:

$$\{x = y, y\ (< \cup >)\ z\}$$

- Semantics: Interpret the time point symbols and relation symbols over the real (or rational) numbers.

Some Reasoning Problems

$$\left\{x\,(<\cup=)\,y, y\,(<\cup=)\,z, v\,(<\cup=)\,y, w>y, z\,(<\cup=)\,x\right\}$$

- Satisfiability: Are there values for all time points such that all formulae are satisfied?
- Satisfiability with $v = w$?
- Finding a satisfying instantiation of all time points
- Deduction: Does $x\{=\}y$ follow logically?
  Does $v \leq w$ follow?
- Finding a minimal description: What are the most constrained relations that describe the same set of instantiations?

From a Logical Point of View . . .

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in satisfiability, model finding, and deduction

Let $\mathcal{B}$ be a finite set of (binary) relations on some (infinite) domain D (elements of $\mathcal{B}$ are called base relations).
We require:

- The relations in $\mathcal{B}$ are JEPD, i.e., jointly exhaustive and pairwise disjoint.
- $\mathcal{B}$ is closed under converses.

Then:

- Let $\mathcal{A}$ be the set of relations that can be built by taking the unions of relations from $\mathcal{B}$ ($\rightsquigarrow 2^{|\mathcal{B}|}$ different relations).
- $\mathcal{A}$ is closed under converse, complement, intersection and union.
- Often, $\mathcal{A}$ is closed under composition of base relations, i.e., for all $B, B' \in \mathcal{B}$,

$$B \circ B' \in \mathcal{A}.$$

  Then, $\mathcal{A}$ is closed under composition of arbitrary relations.

Computing Operations on Relations

Let $\mathcal{A}$ be the system of relations over a set of base relations $\mathcal{B}$ that satisfies all the conditions above.
We may write relations as *sets* of base relations:

$$B_1 \cup \cdots \cup B_n \cong \{B_1, \ldots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \ldots B_n\} \circ \{B'_1, \ldots, B'_m\} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_i \circ B'_j$$

Converse:

$$\{B_1, \ldots, B_n\}^{-1} = \{B_1^{-1}, \ldots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \ldots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and union are defined in the usual set-theoretical way.

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Reasoning Problems

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition xRy: Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y of variables the strongest constrained (minimal) relation entailed by the CSP.

Reductions between CSP Problems

Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.

CSAT $\leq_T$ CENT and CENT $\leq_T$ CMIN are obvious.

CENT $\leq_T$ CSAT: We solve CENT ($CSP \models xRy?$) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let $B_1, \ldots, B_k$ be the relations for which we get a positive answer. Then $x\{B_1, \ldots, B_k\}y$ is entailed by the CSP.

CMIN $\leq_T$ CENT: We use entailment for computing the minimal constraint for each pair of variables. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

The Path Consistency Method

Given a qualitative CSP with $R_{v_1,v_2} = R^{-1}_{v_2,v_1}$. Then the path consistency method is to apply the operation

$$R_{v_1,v_2} \leftarrow R_{v_1,v_2} \cap (R_{v_1,v_3} \circ R_{v_3,v_2}).$$

on all the constraints of the network until a fixpoint is reached.

The path consistency method guarantees ...

- sometimes minimality
- sometimes satisfiability
- however sometimes the CSP is not satisfiable, even if the CSP contains only base relations

Example: Point Relations

Composition table:

	$<$	$=$	$>$
$<$	$<$	$<$	$<, =, >$
$=$	$<$	$=$	$>$
$>$	$<, =, >$	$>$	$>$

Figure: Composition table for the point algebra. For example: $\{<\} \circ \{=\} = \{<\}$

- $\{<, =\} \circ \{<\} = \{<\}$
- $\{<, >\} \circ \{<\} = \{<, =, >\}$
- $\{<, =\}^{-1} = \{>, =\}$
- $\{<, =\} \cap \{>, =\} = \{=\}$

Some Properties of the Point Relations

Theorem

A path consistent CSP over the point relations is satisfiable.

In particular, the path consistency method decides satisfiability.

Theorem

A path consistent CSP over all point relations without $\{<, >\}$ *is minimal.*

Proofs later ...

A Pathological Relation System

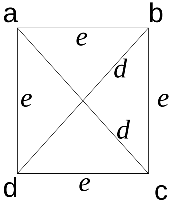
Let e, d, i be (self-converse) base relations between points on a circle:

- e: Rotation by 72 degrees (left or right)
- d: Rotation by 144 degrees (left or right)
- i: Identity

Composition table:

$$e \circ e = \{i, d\}$$
$$d \circ d = \{i, e\}$$
$$e \circ d = \{e, d\}$$
$$d \circ e = \{e, d\}$$

The following CSP is path-consistent and contains only base relations, but it is not satisfiable:



Qualitative Constraint Languages

From now on, let D be a finite or infinite domain.

Definition

A partition scheme on D is any non-empty, finite set Δ of binary relations on D such that:

- Δ defines a partition of $D \times D$.
- Δ contains the binary identity relation id_D.
- Δ is closed under converses.

Definition

A constraint language of binary relations on D, Γ, is said to be generated from a partition scheme Δ, if Γ consists of all finite unions of relations in Δ.

Constraint languages in this sense will be referred to as qualitative constraint languages.

Qualitative Constraint Network

Let Γ be a subset of a qualitative constraint language with partition scheme Δ.

Definition

A qualitative constraint network over Γ is a triple

$$P = \langle V, D, C \rangle ,$$

where:

- V is a non-empty and finite set of variables,
- D is an arbitrary non-empty set (domain),
- C is a finite set of constraints $C_1, \ldots, C_q$, i.e., each constraint C_i is a pair (s_i, R_i), where s_i is a pair of variables and R_i is a binary relation contained in Γ.

Weak Composition

Let Γ be a qualitative constraint language with partition scheme Δ. For $R, S \in \Gamma$, define:

$$R \circ_w S := \bigcup \{T \in \Delta : T \cap (R \circ S) \neq \emptyset\}$$

— $\circ_w$ is called weak composition of R and S.

Lemma

For all relations $R, S, T \in \Gamma$,

- $R \circ S \subseteq R \circ_w S$;
- $T \cap (R \circ S) = \emptyset$ *if and only if* $T \cap (R \circ_w S) = \emptyset$;
- $(R \circ_w S)^{-1} = S^{-1} \circ_w R^{-1}$;
- $R \circ_w (S \cup T) = (R \circ_w S) \cup (R \circ_w T)$.

Weak Composition: Examples

Example:
Consider a linear order on a domain with 2 elements $a < b$. The relations $R_<, R_=, R_>$ define a partition schema on D. It holds:

$$R_< \circ R_< = R_> \circ R_> = \emptyset, \;\; R_< \circ R_> = \{(a,a)\}, \;\; R_> \circ R_< = \{(b,b)\}$$

but

$$R_< \circ_w R_< = R_> \circ_w R_> = \emptyset, \;\; R_< \circ_w R_> = R_=, \;\; R_> \circ_w R_< = R_=$$

Moreover,

$$(R_< \circ_w R_>) \circ_w R_> = R_= \circ_w R_> = R_> \neq \emptyset = R_< \circ_w \emptyset = R_< \circ_w (R_> \circ_w R_>).$$

Example:
Consider a linear order on a domain with 3 elements $a < b < c$. Then

$$R_< \circ R_< = \{(a,c)\} \quad \text{but} \quad R_< \circ_w R_< = R_<.$$

Non-Associative Relation Algebras

Definition

A non-associative relation algebra is a set A with

- binary operations $\sqcap$, $\sqcup$, and $;$,
- unary operations $-$ and $^{\smile}$, and
- distinct elements 0, 1, and δ such that

(a) $(A, \sqcap, \sqcup, -, 0, 1)$ is a Boolean algebra.

(b) For all elements a, b and c of A:

$$a\,;(b \sqcup c) = (a\,;b) \sqcup (a\,;c)$$

$$\delta\,;a = a\,;\delta = a$$

$$(a^{\smile})^{\smile} = a \text{ and } (-a)^{\smile} = -(a^{\smile})$$

$$(a \sqcup b)^{\smile} = a^{\smile} \sqcup b^{\smile}$$

$$(a\,;b)^{\smile} = b^{\smile}\,;a^{\smile}$$

$$(a\,;b) \sqcap c^{\smile} = 0 \text{ if and only if } (b\,;c) \sqcap a^{\smile} = 0$$

Qualitative Languages and Algebras

Let Γ be a qualitative constraint language with partition scheme Δ. As spelled out before, each relation R in Γ can be represented by a finite disjunction of "base relations" $B_1, \ldots, B_k \in \Delta$. In what follows we identify R with the set of its base relations

$$\{B_1, \ldots, B_k\}.$$

Lemma

For each partition scheme Δ, the tuple

$$\left\langle 2^{\Delta}, \cap, \cup, \circ_w, \mathbf{C}_{\Delta}, {}^{-1}, \emptyset, \Delta, \mathrm{id}_{\Delta} \right\rangle$$

defines a non-associative relation algebra.

Algebraically Closed Networks

A qualitative network $P = \langle V, D, C \rangle$ is normalized, if

- for each pair of variables x, y, C contains at least one constraint $((x, y), R)$;
- for each constraint $((x, x), R)$ in C, $R = \mathrm{id}_D$;
- for constraints $((x, y), R)$ and $((y, x), S)$ in C, $R = S^{-1}$.

In what follows we will always assume that constraint networks are normalized.

Definition

A qualitative constraint network P is algebraically closed (or: a-closed), if for all constraints $((x, y), R)$, $((x, z), S)$, and $((z, y), T)$ of P, it holds:

$$R \subseteq S \circ_w T.$$

Note: If P is algebraically closed, then $R = R \cap (S \circ_w T)$.

Constraint Propagation

The path consistency algorithm can only be used if the underlying partition scheme is closed under composition, i.e., if for each pair of relations $R, S \in \Delta$, $R \circ S$ is a (finite) union of a subset of Δ.

The algebraic closure algorithm is a variant of the path consistency algorithm. Instead of ordinary composition of relations, we use weak composition.

Since weak composition is an upper approximation of composition only, the algebraic closure algorithm may not result in a path-consistent network.

Let $P = \langle V, D, C \rangle$ be a (normalized) qualitative constraint network.

Let *Table*$[i, j]$ be a $n \times n$-matrix (n: number of variables), in which we record the constraints between the variables.

Algebraic Closure Algorithm

EnforceAlgClosure (P):
Input: a qualitative network $P = \langle V, D, C \rangle$
Output: "inconsistent", or an equivalent algebraically closed network P'

$$\mathit{Paths}(i,j) = \{(i,j,k) : 1 \leq k \leq n, k \neq i,j\} \cup \{(k,i,j) : 1 \leq k \leq n, k \neq i,j\}$$

$\mathit{Queue} := \bigcup_{i,j} \mathit{Paths}(i,j)$

while $Q \neq \emptyset$
  select and delete (i,k,j) from Q
  $T := \mathit{Table}[i,j] \cap (\mathit{Table}[i,k] \circ_w \mathit{Table}[k,j])$
  if $T = \emptyset$
    return "inconsistent"
  elseif $T \neq \mathit{Table}[i,j]$
    $\mathit{Table}[i,j] := T$
    $\mathit{Table}[j,i] := T^{-1}$
    $\mathit{Queue} := \mathit{Queue} \cup \mathit{Paths}(i,j)$
return P' with the refined constraints as recorded in *Table*

Computing on the Symbolic Level

Let Γ be a qualitative constraint language with partition scheme Δ.
We suppose that we have determined (by some formal proof or some computation) the (weak) composition table for Δ, i.e.,

$$\circ_{(w)} \colon \Delta \times \Delta \to 2^{\Delta}.$$

Let now B be a finite set of symbols (bijective with Δ).
Then 2^B is a Boolean algebra, from which we obtain a (non-associative) relation algebra, if we extend $\circ_{(w)}$ to a function

$$\circ_{(w)} : 2^B \times 2^B \to 2^B.$$

Now we can perform all the operations needed in the path consistency/a-closure algorithm on the symbolic level.

Path Consistency and Tractability

Let Γ be a subset of a qualitative constraint language with a partition scheme Δ that is closed under composition.
Let $\widehat{\Gamma}$ be smallest superset of Γ that is closed under intersection, converses, and composition.

Lemma

There exists a polynomial time reduction from CSP$(\widehat{\Gamma})$ to CSP(Γ), provided Γ contains identity and the universal relation. In particular, it holds:

- *Γ is tractable if and only if $\widehat{\Gamma}$ is tractable.*
- *Enforcing path consistency decides satisfiability over $\widehat{\Gamma}$ if and only if it does so over Γ.*

Proof idea.

Each relation in $\widehat{\Gamma}$ stems from a finite number of compositions, intersections, and conversions applied to relations in Γ. Hence each constraint network over $\widehat{\Gamma}$ can be transformed step-by-step into an equivalent network over Γ. In the case where a relation results from composing other relations, we need to introduce some fresh variables. □

Algebraic Closure and Tractability

Let now Γ be a subset of a qualitative constraint language with a partition scheme Δ (not necessarily closed under composition).

Let $\widehat{\Gamma}^w$ be smallest superset of Γ that is closed under intersection, converses, and weak composition.

Lemma (Ligozat & Renz 2005)

If enforcing a-closure decides satisfiability for atomic networks (i.e., for qualitative networks over Δ), then CSP$(\widehat{\Gamma}^w)$ is polynomial-time reducible to CSP(Γ) (provided Γ contains idendentity and the universal relation.

In particular, if a-closure decides satisfiability for atomic networks, then

- *Γ is tractable if and only if $\widehat{\Gamma}^w$ is so;*
- *enforcing a-closure decides satisfiability over Γ if and only if a-closure decides satisfiability over $\widehat{\Gamma}^w$.*

Allen's Interval Calculus

- Allen's interval calculus (IA): time intervals and binary relations over them
- Let $\langle \mathbb{R}, < \rangle$ be the linear order on the real numbers (conceived of as the flow of time).
  Then, the domain D of Allen's calculus is the set of all *intervals*

  $$X = (X^-, X^+) \in \mathbb{R}^2, \text{ where } X^- < X^+$$

  (naïve approach)
- Relations between concrete intervals, e. g.:

  (1.0,2.0) strictly before (3.0,5.5)
  (1.0,3.0) meets (3.0,5.5)
  (1.0,4.0) overlaps (3.0,5.5)
  ...

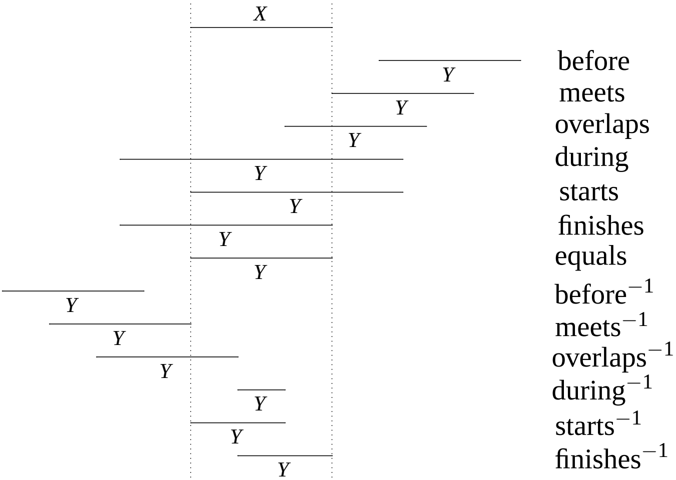
IA: The Base Relations

To determine all possible relation between Allen intervals, we determine how one can order the four points of two intervals:

Relation	Symbol	Name
$\{(X,Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X,Y) : X^- < X^+ = Y^- < Y^+\}$	$\mathtt{m}$	meets
$\{(X,Y) : X^- < Y^- < X^+ < Y^+\}$	$\mathtt{o}$	overlaps
$\{(X,Y) : X^- = Y^- < X^+ < Y^+\}$	$\mathtt{s}$	starts
$\{(X,Y) : Y^- < X^- < X^+ = Y^+\}$	$\mathtt{f}$	finishes
$\{(X,Y) : Y^- < X^- < X^+ < Y^+\}$	$\mathtt{d}$	during
$\{(X,Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the *converse* relations (obtained by exchanging X and Y)

IA: The 13 Base Relations Graphically



IA: Partition Scheme and Composition

Lemma

The 13 base relations of Allen's interval calculus define a partition scheme on the set of all Allen intervals.

In what follows:

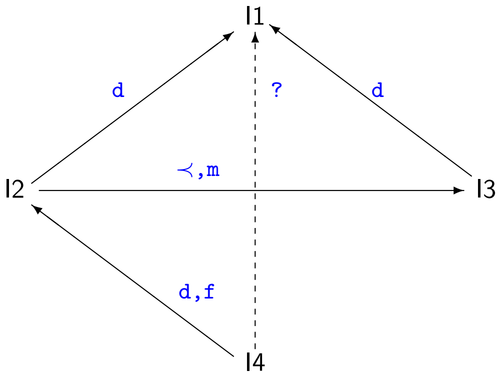
- IA: the qualitative constraint language generated from all base relations of Allen's interval calculus (contains $2^{13} = 8192$ relations)
- IA-$\mathcal{B}$: the subclass of IA containing base relations only

Lemma

The set of base relations of Allen's interval calculus is closed under composition.

	$\prec$	$\succ$	d	d^{-1}	o	o^{-1}	m	m^{-1}	s	s^{-1}	f	f^{-1}
$\prec$	$\prec$	**B**	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$	$\prec$ o m d s	$\prec$	$\prec$	$\prec$ o m d s	$\prec$
$\succ$	**B**	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$ o^{-1} m^{-1} d f	$\succ$	$\succ$	$\succ$
d	$\prec$	$\succ$	d	**B**	$\prec$ o m d s	$\succ$ o^{-1} m^{-1} d f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d f	d	$\prec$ o m d s
d^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	$\mathbf{B}-$ $\prec\succ$ m m^{-1}	d^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	d^{-1}	o^{-1} d^{-1} s^{-1}	d^{-1}
o	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	$\mathbf{B}-$ $\prec\succ$ m m^{-1}	$\prec$	o^{-1} d^{-1} s^{-1}	o	d^{-1} f^{-1} o	d s o	$\prec$ o m
o^{-1}	$\prec$ o m, d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$, o^{-1} m^{-1} d^{-1} s^{-1}	$\mathbf{B}-$ $\prec\succ$ m m^{-1}	$\succ$ o^{-1} m^{-1}	o d^{-1} f^{-1}	$\succ$	o^{-1} d f	o^{-1} $\succ$ m^{-1}	o^{-1}	o^{-1} d^{-1} s^{-1}
m	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\prec$	$\prec$	o d s	$\prec$	f f^{-1} $\equiv$	m	m	d s o	$\prec$
m^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} d f	$\succ$	o^{-1} d f	$\succ$	s s^{-1} $\equiv$	$\succ$	d f o^{-1}	$\succ$	m^{-1}	m^{-1}
s	$\prec$	$\succ$	d	$\prec$ o m d^{-1} f^{-1}	$\prec$ o m	o^{-1} d f	$\prec$	m^{-1}	s	s s^{-1} $\equiv$	d	$\prec$ m o
s^{-1}	$\prec$ o m d^{-1} f^{-1}	$\succ$	o^{-1} .d f	d^{-1}	o d^{-1} f^{-1}	o^{-1}	o d^{-1} f^{-1}	m^{-1}	s s^{-1} $\equiv$	s^{-1}	o^{-1}	d^{-1}
f	$\prec$	$\succ$	d	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	$\succ$ o^{-1} m^{-1}	m	$\succ$	d	$\succ$ o^{-1} m^{-1}	f	f f^{-1} $\equiv$
f^{-1}	$\prec$	$\succ$ o^{-1} m^{-1} d^{-1} s^{-1}	o d s	d^{-1}	o	o^{-1} d^{-1} s^{-1}	m	s^{-1} o^{-1} d^{-1}	o	d^{-1}	f f^{-1} $\equiv$	f^{-1}

IA: An Example



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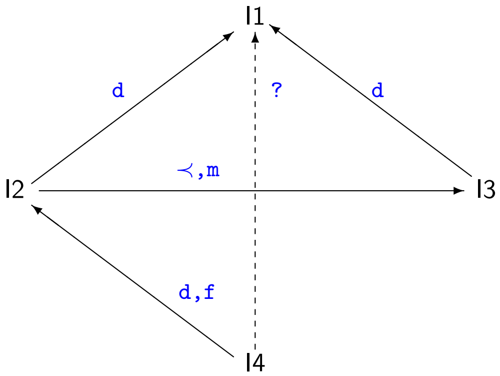

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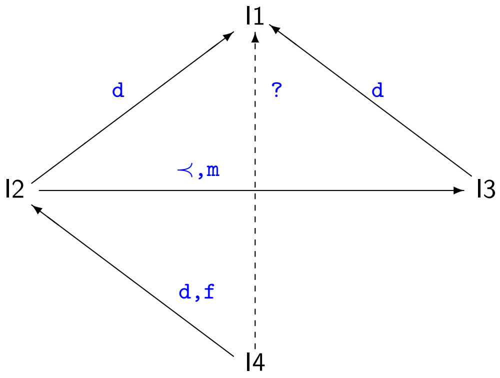

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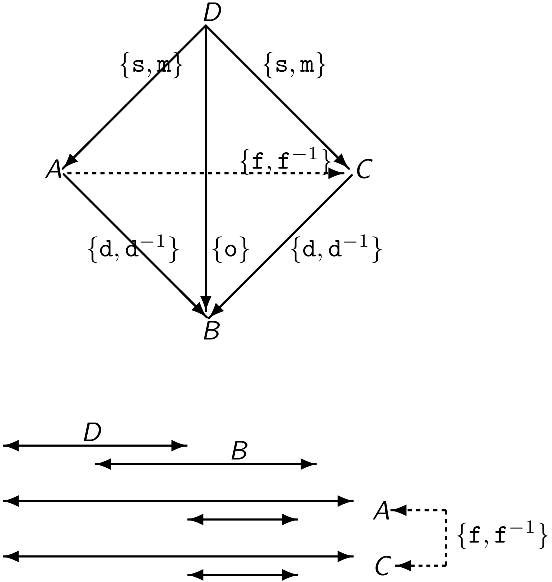
IA: Example for Incompleteness



IA: NP-Hardness

Theorem (Kautz & Vilain)

Deciding satisfiability over IA is NP-hard.

Proof.

Reduction from 3-colorability (the original proof uses 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

$$
\begin{array}{cccl}
1 & \{\mathtt{m}\} & 2 & \\
2 & \{\mathtt{m}\} & 3 & \\
v_i & \{\mathtt{m}, \equiv, \mathtt{m}^{-1}\} & 2 & \forall v_i \in V \\
v_i & \{\mathtt{m}, \mathtt{m}^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E
\end{array}
$$

This constraint system is satisfiable *iff* G can be colored with 3 colors. □

IA: Clause Representation

Following, we will look at polynomial special cases, i.e., subclasses of the qualitative constraint language IA.

For this we start from a natural translation of interval relations/constraints (of the form $X\,R\,Y$) into clause formulas over atoms of the form $a\;op\;b$, where:

- $a, b \in \{X^-, X^+, Y^-, Y^+\}$; and
- $op \in \{<, >, =, \leq, \geq\}$.

Example: All base relations can be expressed as unit clauses.

Lemma

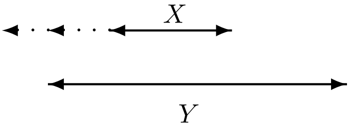
Let P be a constraint network over IA, and let $\pi(P)$ be the translation of P into clause form.
P is satisfiable iff $\pi(P)$ is satisfiable over the real numbers.

IA: The Continuous Endpoint Class

Continuous Endpoint Class IA-$\mathcal{C}$: the subset of IA consisting of those relations with a clause form containing only unit clauses, where $\neg(a = b)$ is forbidden.

Example: All basic relations and, e.g., $\{\mathtt{d}, \mathtt{o}, \mathtt{s}\}$, because

$$\begin{aligned}\pi(X\,\{\mathtt{d},\mathtt{o},\mathtt{s}\}\,Y) \quad = \quad \{\,& X^- < X^+, Y^- < Y^+,\\ & X^- < Y^+, X^+ > Y^-,\\ & X^+ < Y^+\}\end{aligned}$$



The set IA-$\mathcal{C}$ contains 83 relations. It is closed under intersection, composition, and converses (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

IA: Consistency for IA-$\mathcal{C}$

Following we prove:

Lemma

Each 3-consistent interval CSP over IA-$\mathcal{C}$ is globally consistent.

From this we can conclude:

Theorem (van Beek)

Applied to networks over IA-$\mathcal{C}$, enforcing path consistency decides satisfiability and solves the minimal label problem.

Corollary

A path-consistent interval constraint network containing base relations only is satisfiable.

Helly's Theorem

Definition

A set $M \subseteq \mathbb{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M.

Theorem (Helly)

Let F be a family of at least $n+1$ convex sets in $\mathbb{R}^n$. If all sub-families of F with $n+1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

IA: Strong n-Consistency (1)

Proof of the lemma.

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \surd$

Induction assumption: Assume strong $k-1$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k-1$ variables X_i to pairs $(s_i, e_i) \in \mathbb{R}^2$ satisfying the constraints R_{ij} between the $k-1$ variables.

We have to show that we can extend the instantiation to any kth variable.

IA: Strong n-Consistency (2): Instantiating the kth Variable

Proof (Part 2).

The instantiation of the $k-1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k.

Note: Since $R_{ij} \in$ IA-$\mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \ldots$$

Such inequalities define convex subsets in $\mathbb{R}^2$.

$\rightsquigarrow$ Consider sets of 3 inequalities (= 3 convex sets).

IA: Strong n-Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+, but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by the same argument as above there exists a common point.

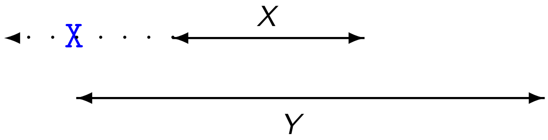
- $\rightsquigarrow$ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.
- $\rightsquigarrow$ Strong k-consistency for all $k \leq n$.

IA: The Endpoint Subclass

Endpoint Subclass: IA-$\mathcal{P}$ is the subclass that permits a clause form containing only unit clauses ($a \neq b$ is now allowed).

Example: all basic relations and $\{\mathtt{d}, \mathtt{o}\}$ since

$$\pi(X\ \{\mathtt{d}, \mathtt{o}\}\ Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, X^+ < Y^+\}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path consistency method decides satisfiability over IA-$\mathcal{P}$.

IA: The ORD-Horn Subclass

ORD-Horn Subclass: IA-$\mathcal{H}$ is the subclass of IA that permits a clause form containing only Horn clauses, where only the following literals are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in$ IA-$\mathcal{P}$ and $\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}$:

$$\pi(X\{\mathtt{o}, \mathtt{s}, \mathtt{f}^{-1}\}Y) = \begin{cases} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{cases}.$$

Partial Orders: The ORD Theory

Let ORD be the following theory:

$$\begin{array}{llcll}
\forall x, y, z\colon & x \leq y \wedge y \leq z & \rightarrow & x \leq z & \textit{(transitivity)} \\
\forall x\colon & x \leq x & & & \textit{(reflexivity)} \\
\forall x, y\colon & x \leq y \wedge y \leq x & \rightarrow & x = y & \textit{(anti-symmetry)} \\
\forall x, y\colon & x = y & \rightarrow & x \leq y & \textit{(weakening of } =) \\
\forall x, y\colon & x = y & \rightarrow & y \leq x & \textit{(weakening of } =).
\end{array}$$

- ORD describes partially ordered sets, $\leq$ being the ordering relation.
- ORD is a Horn theory
- What is missing wrt. *dense* and *linear* orders?

Satisfiability over Partial Orders

Lemma

Let Θ be a CSP over IA-$\mathcal{H}$. Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i. e., satisfy ORD), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

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Complexity of CSAT(IA-$\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Lemma

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Theorem

CSAT(IA-$\mathcal{H}$) can be decided in polynomial time.

Proof.

CSAT(IA-$\mathcal{H}$) instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. □

IA-$\mathcal{C}$ $\subset$ IA-$\mathcal{P}$ $\subset$ IA-$\mathcal{H}$ with $|$IA-$\mathcal{C}| = 83$, $|$IA-$\mathcal{P}| = 188$, $|$IA-$\mathcal{H}| = 868$

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Path Consistency and the OH-Class

Lemma

Let Θ be a path-consistent set over IA-$\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \quad \textit{iff} \quad \Theta \textit{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides CSAT(IA-$\mathcal{H}$).

- ⇝ Maximality of IA-$\mathcal{H}$?
- ⇝ Do we have to check all $8192 - 868$ extensions?

IA: The ORD-Horn Subclass: Maximality

A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras containing all base relations that strictly contain IA-$\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{\mathtt{d}, \mathtt{d}^{-1}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}\} \in \mathcal{X}_1$$
$$N_2 = \{\mathtt{d}^{-1}, \mathtt{o}, \mathtt{o}^{-1}, \mathtt{s}^{-1}, \mathtt{f}^{-1}\} \in \mathcal{X}_2$$

The clause forms of these relations contain "proper" disjunctions!

Theorem

The satisfiability problem over IA-$\mathcal{H} \cup \{N_i\}$ is NP-complete.

Lemma

IA-$\mathcal{H}$ is the only maximal tractable subclass that contains all base relations of IA.

IA: Solving General Allen CSPs

- Backtracking algorithm using path consistency as a forward-checking method
- Method works on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

⇝ Which tractable fragment should one use?

IA: Branching Factors

- If the labels are split into base relations, then on average a label is split into

6.5 relations

- If the labels are split into pointizable relations ($\mathcal{P}$), then on average a label is split into

2.955 relations

- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into

2.533 relations

$\rightsquigarrow$ A difference of 0.422 which becomes significant, when applied to extremely hard instances

RCC8: Motivation

We may want to state qualitative relationships between regions in space, for example:

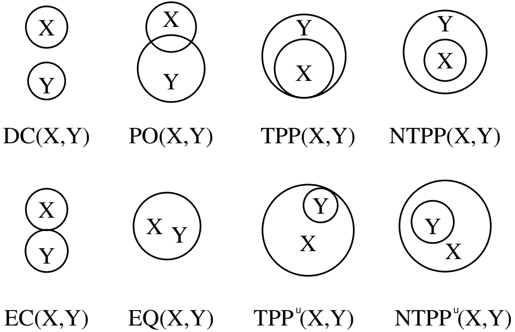
- "Region X touches region Y"
- "Germany and Switzerland have a common border"
- "Freiburg is located in Baden-Württemberg"

RCC8: Possible Applications

- This can be useful when only partial information is available:
  - We may know that region X is not connected with region Y without knowing the shape and location of X and Y.
- We may want to query a database:
  - Show me all countries bordering the Mediterranean!
- We may want to state integrity constraints:
  - An island has to be located in the interior of a sea.

RCC8: Qualitative Relations Between Regions

Eight relations between regions:



RCC8: Intuition

- Regions are some "reasonable" non-empty subsets of space.
- DC (disconnected) means that the two regions do not share any point at all.
- EC (externally connected) means that they only share borders.
- PO (partially overlapping) means that the two regions share interior points.
- TPP (tangential proper part) means that one region is a subset of the other sharing some points on the borders.
- NTPP (non-tangential proper part) same, but without sharing any bordering points.

Point-Set Topology

Point-set topology is a mathematical theory that deals with properties of space independent of size and shape.

In topology, we can define notions such as

- interior and exterior points of regions,
- isolated points of regions,
- boundaries of regions,
- connected components of regions,
- connected regions,
- . . .

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- connected components of regions,
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- ...

Topology

Definition

A topological space is a pair $\mathcal{T} = (S, \mathcal{O})$, where

- S is a non-empty set (the universe), and
- $\mathcal{O}$ is a set of subsets of S (the open sets)

such that the following conditions hold:

- $\emptyset \in \mathcal{O}$ and $S \in \mathcal{O}$.
- If $O_1 \in \mathcal{O}$ and $O_2 \in \mathcal{O}$, then $O_1 \cap O_2 \in \mathcal{O}$.
- If $(O_i)_{i \in I}$ is a (possibly infinite) family of elements from $\mathcal{O}$, then

$$\bigcup_{i \in I} O_i \in \mathcal{O}.$$

Example: In Euclidean space, a set O is open if for each point $x \in O$ there is a ball surrounding x that is contained in O.

Terminology & Notation

Definition

Let $X \subseteq S$ and $x \in S$.

- A set $N \subseteq S$ is a neighborhood of a point x if there is an open set $O \in \mathcal{O}$ such that $x \in O \subseteq N$.
- $x \in S$ is an interior point of X if there is a neighborhood N of x such that $N \subseteq X$.
- $x \in S$ is a touching point of X if every neighborhood of x has a non-empty intersection with X.

Notation:

- $\text{int}(X)$ is the set of interior points of X (the interior of X).
- $\text{cls}(X)$ is the set of touching points of X (the closure of X).
- A set is closed if $X = \text{cls}(X)$.

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Interior and Closure Operators

The function $\text{int}(\cdot)$ is an interior operator:

1. $\text{int}(S) = S$
2. $\text{int}(X) \cap \text{int}(Y) = \text{int}(X \cap Y)$
3. $\text{int}(X) \subseteq X$
4. $\text{int}(\text{int}(X)) = \text{int}(X)$

Note:

- X is *open* iff $X = \text{int}(X)$
- $\text{cls}(X) = S \setminus \text{int}(S \setminus X)$

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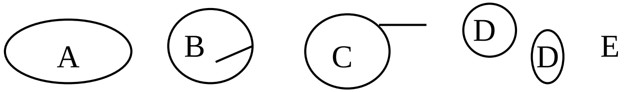
RCC8: What Is a Region?



A and D are reasonable regions, B, C, and E are not

In other words, X is a region iff it is non-empty

$$X \neq \emptyset$$

and regular closed, i. e., the closure of an open set:

$$X = \mathsf{cls}(\mathsf{int}(X)).$$

It is not necessary that a region is internally connected.

Defining the RCC8-Relations

Let S be a topological space. Then define the following relations on Reg:

$$\begin{aligned}
\mathsf{DC}(X,Y) &:= X \cap Y = \emptyset \\
\mathsf{EC}(X,Y) &:= X \cap Y \neq \emptyset \wedge \mathsf{int}\, X \cap \mathsf{int}\, Y = \emptyset \\
\mathsf{PO}(X,Y) &:= \mathsf{int}\, X \cap \mathsf{int}\, Y \neq \emptyset \wedge X \not\subseteq Y \wedge Y \not\subseteq X \\
\mathsf{EQ}(X,Y) &:= X = Y \\
\mathsf{TPP}(X,Y) &:= X \subseteq Y \wedge X \not\subseteq \mathsf{int}\, Y \\
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\end{aligned}$$

$\rightsquigarrow$ It can be seen that these relations define a partition scheme.

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RCC8: From Regions to Boolean Algebras

Let Reg denote the set of all regular closed set of some fixed topological space.
For $X, Y \in \mathsf{Reg} \cup \{\emptyset\}$ define:

$$-X := \mathsf{cls}(S \setminus X)$$
$$X \sqcup Y := X \cup Y$$
$$X \sqcap Y := \mathsf{cls}(\mathsf{int}(X \cap Y))$$

By these definition, we obtain a Boolean algebra.

Boolean Connection Algebras

Definition

A connection algebra is a Boolean algebra B together with a binary relation C on B such that the following conditions are satisfied:

- $x \neq 0 \Leftrightarrow x\ C\ x$
- $x\ C\ y \Rightarrow y\ C\ x$
- $x \neq 0, 1 \Rightarrow x\ C\ -x$
- $x\ C\ y \cup z \Leftrightarrow x\ C\ y$ or $x\ C\ z$
- $x \neq 0, 1 \Rightarrow$ not $x\ C\ y$, for some $y \neq 0, 1$

RCC8: From Topologies to Connection Algebras

If the underlying topological space is regular and connected, i.e.,

- Hausdorff and for each $x \in S$ and closed subset $A \subseteq S$ with $x \notin A$, there exist disjoint open neighborhoods of x and A;
- the only sets that are open and closed are $\emptyset$ and S;

then

$$x \mathrm{C} y \iff x \cap y \neq \emptyset$$

defines a connection algebra on Reg $\cup \{\emptyset\}$.

Defining the RCC8-Relations (2)

Let B be a connection algebra. Then we can define the RCC8 relations on $B \setminus \{0\}$ as follows:

$$
\begin{aligned}
X \ \mathrm{DC} \ Y &:= \text{not } X \ \mathrm{C} \ Y \\
X \ \mathrm{P} \ Y &:= (X,Y) \notin \mathrm{C} \circ \mathrm{DC} \\
X \ \mathrm{PP} \ Y &:= X \ \mathrm{P} \ Y \wedge X \neq Y \\
X \ \mathrm{O} \ Y &:= (X,Y) \in \mathrm{P}^{-1} \circ \mathrm{P} \\
X \ \mathrm{PO} \ Y &:= X \ \mathrm{O} \ Y \wedge \text{not } X \ \mathrm{P} \ Y \wedge \text{not } Y \ \mathrm{P} \ X \\
X \ \mathrm{EC} \ Y &:= X \ \mathrm{C} \ Y \wedge \text{not } X \ \mathrm{O} \ Y \\
X \ \mathrm{TPP} \ Y &:= X \ \mathrm{PP} \ Y \wedge (X,Y) \in \mathrm{EC} \circ \mathrm{EC} \\
X \ \mathrm{NTPP} \ Y &:= X \ \mathrm{PP} \ Y \wedge \text{not } X \ \mathrm{TPP} \ Y \\
&\ldots
\end{aligned}
$$

RCC8: Complexity

Using a reduction from 3SAT, it can be shown:

Theorem

Testing satisfiability over arbitrary RCC8 relations is NP-hard.

Using a translation into S4-modal logics, one can show:

Theorem

Testing satisfiability over arbitrary RCC8 relations is NP-complete.

Lower Bound: Proving NP-Hardness

- Idea: Reduction from 3-SAT
- 3-SAT structure
  1. Literals a, b, c: can be true or false
  2. Complementary literals: a is true iff $\neg a$ is false
  3. Clauses $l_1 \vee l_2 \vee l_3$: at least one literal must be true
- RCC8-CSP
  1. Truth value constraints $X_a\{R_t, R_f\}Y_a$: Either $X_a\{R_t\}Y_a$ or $X_a\{R_f\}Y_a$ holds
  2. Polarity constraints: $X_a\{R_t\}Y_a$ holds iff $X_{\neg a}\{R_f\}Y_{\neg a}$ holds
  3. Clause constraints: At least one of $X_{l_1}\{R_t\}Y_{l_1}$, $X_{l_2}\{R_t\}Y_{l_2}$, or $X_{l_3}\{R_t\}Y_{l_3}$ holds

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- RCC8-CSP
  1. Truth value constraints $X_a\{R_t, R_f\}Y_a$: Either $X_a\{R_t\}Y_a$ or $X_a\{R_f\}Y_a$ holds
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Lower Bound: Proving NP-Hardness

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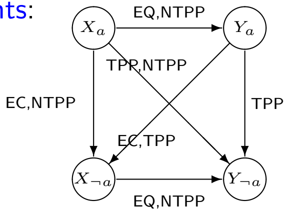
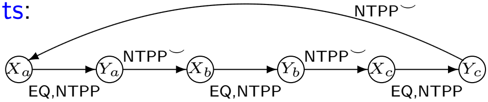
The Reduction

- Relations: $R_t = NTPP$, $R_f = EQ$
- Polarity constraints:



- Clause constraints:



- RCC8 sat.⇒3-SAT: follows from reduction
- 3-SAT⇒RCC8 sat.: Construction of model for Θ_ϕ for each positive 3-SAT instance ϕ

RCC8: Constraint Propagation

- As in Allen's interval algebra, we may want to use constraint propagation instead of translating everything to modal logic.
- We need a composition table . . .
- . . . which could be computed using the modal logic encoding (and in fact, this has been done).
- Based on this table, we can then apply the algebraic closure algorithm
- . . . and ask ourselves for which fragment of RCC8 it is complete.

$\circ$	DC	EC	PO	TPP	NTPP	TPP^{-1}	$NTPP^{-1}$	EQ
DC	*	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC,EC PO,TPP NTPP	DC	DC	DC
EC	DC,EC PO,TPP^{-1} $NTPP^{-1}$	DC,EC PO,TPP TPP^{-1},EQ	DC,EC PO,TPP NTPP	EC,PO TPP NTPP	PO TPP NTPP	DC,EC	DC	EC
PO	DC,EC PO,TPP^{-1} $NTPP^{-1}$	DC,EC PO,TPP^{-1} $NTPP^{-1}$	*	PO TPP NTPP	PO TPP NTPP	DC,EC PO, TPP^{-1} $NTPP^{-1}$	DC,EC PO,TPP^{-1} $NTPP^{-1}$	PO
TPP	DC	DC,EC	DC,EC PO,TPP NTPP	TPP NTPP	NTPP	DC,EC PO,TPP TPP^{-1},EQ	DC,EC PO,TPP^{-1} $NTPP^{-1}$	TPP
NTPP	DC	DC	DC,EC PO,TPP NTPP	NTPP	NTPP	DC,EC PO,TPP NTPP	*	NTPP
TPP^{-1}	DC,EC PO,TPP^{-1} $NTPP^{-1}$	EC,PO TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO,EQ TPP TPP^{-1}	PO TPP NTPP	TPP^{-1} $NTPP^{-1}$	$NTPP^{-1}$	TPP^{-1}
$NTPP^{-1}$	DC,EC PO,TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO TPP^{-1} $NTPP^{-1}$	PO,TPP^{-1} TPP,NTPP $NTPP^{-1}$,EQ	$NTPP^{-1}$	$NTPP^{-1}$	$NTPP^{-1}$
EQ	DC	EC	PO	TPP	NTPP	TPP^{-1}	$NTPP^{-1}$	EQ

RCC8: Is the Composition Table Extensional?

It can easily be verified that already in the 2-dimensional case, the set of base relations is not closed under composition:

- Consider EC $\circ$ TPP and X NTPP S, where S denotes the universal region.
- Consider EC $\circ$ EC and a donut-like region X with "hole" Y.

Lemma (Düntsch et al. 2001)

In each connection algebra, the relation algebra generated by the RCC8 base relations contains at least 25 atomic relations.

Lemma (Li et al. 2006)

In each model associated to some Euclidean space $\mathbb{R}^n$, the relation algebra generated by the RCC8 base relations contains an infinite strictly decreasing sequence of relations.

RCC8: Tractable Fragments?

Theorem (Li 2006)

Enforcing algebraic closure on atomic RCC8 constraint network decides satisfiability.

- As in the case of Allen's interval calculus, we may ask for maximal tractable subsets ...
- Again, one can identify relations that can be encoded by Horn formulae ...
- 148 Horn relations $\mathcal{H}_8$, which forms again a maximal subset.
- There are 2 additional maximal subsets that allow for poly. satisfiability testing!

Some Experiments

- How difficult is the RCC8 satisfiability problem in practice?
- Are there particularly difficult instances?
  - ⇝ Where is the phase transition region?
  - ⇝ Cheeseman et al [IJCAI 91] conjectured that for all NP-complete problems there exists a parameter such that when changing this parameter there exists a very small range – the phase transition region – where the probability of satisfiability of randomly generated instances changes from 1 to 0. They also conjectured that in this area one finds many hard instances.
- How well does the path consistency method approximate satisfiability?
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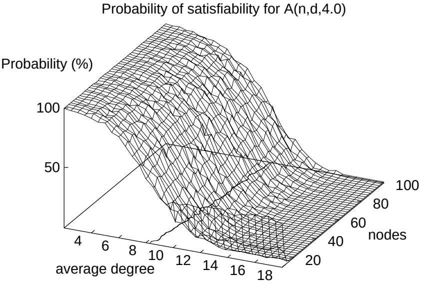
Generating Instances

- Randomly generating instances according to the following parameters:
  - Number of nodes n
  - Average number of constraints d: $(nd/2$ out of $n(n-1)/2$ possible constraints
  - Average number of base relations l per constraint
  - Allowed constraints
    - $A(n,d,l)$: all RCC8 relations
    - $H(n,d,l)$: only relations out of RCC8 $-\ \mathcal{H}_8$

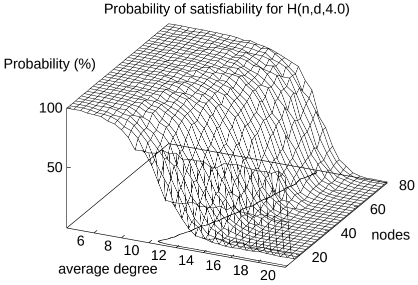
Phase Transition for $A(n, d, 4)$



500 instances per data point

- Phase transition for $A(n, d, 4)$ between $d = 8$ and $d = 10$ for $10 \leq n \leq 100$.

Phase Transition for $H(n, d, 4)$

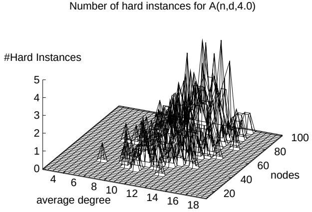


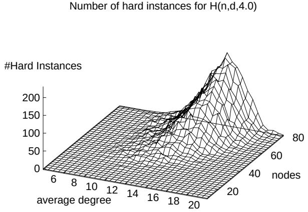
500 instances per data point

- Phase transition for $H(n, d, 4)$ between $d = 10$ and $d = 15$ for $10 \leq n \leq 80$.

Hard Instances ...

... using more than 10,000 search nodes





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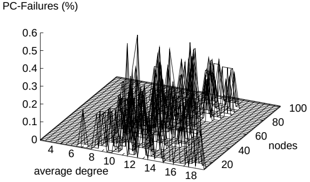
Quality of Path Consistency...

...measured as the percentage of path consistent but unsatisfiable CSPs

Percentage points of incorrect PCA answers for A(n,d,4.0)



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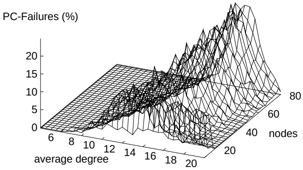

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