

# Constraint Satisfaction Problems

## Qualitative Representation and Reasoning

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# Constraint Satisfaction Problems

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## Quantitative vs. Qualitative Representations

**Spatio-temporal configurations** can be described **quantitatively** by specifying the coordinates of the relevant objects:

**Example:** At time point 10.0 object *A* is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object *B* is at position (15.2, 3.5, 23.7). Object *C* is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).

Often, however, a **qualitative** description (using a finite vocabulary) is more adequate:

**Example:** Object *A* hit object *B*. Afterwards, object *C* arrived.

Sometimes we want to reason with such descriptions.

**Example:** Object *C* was not close to object *A*, when it hit object *B*.

Motivation

## Representation of Qualitative Knowledge

**Intention:** describe configurations in an infinite (continuous) domain using a finite vocabulary and reason about these descriptions

- ▶ Specification of a **vocabulary**: usually a finite set of relations (often binary) that are **pairwise disjoint** and **jointly exhaustive**
- ▶ Specification of a **language**: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- ▶ Specification of a formal **semantics**
- ▶ Analysis of computational properties and design of **reasoning methods** (often constraint propagation)
- ▶ Perhaps, specification of **operational semantics** for verifying whether a relation holds in a given quantitative configuration

## Applications in ...

- ▶ Natural language processing
- ▶ Specification of abstract spatio-temporal configurations
- ▶ Query languages for spatio-temporal information systems
- ▶ Layout descriptions of documents (and learning of such layouts)
- ▶ Action planning
- ▶ ...

## Example: Qualitative Temporal Relations

Suppose, we want to talk about **time instants** (points) and binary **relations** over them.

- ▶ **Vocabulary:**  $X = Y$  ( $X$  equals  $Y$ ),  $X < Y$  ( $X$  before  $Y$ ), and  $X > Y$  ( $X$  after  $Y$ ).
- ▶ **Language:**
  - ▶ Allow for **disjunctions** of basic relations to express **indefinite information**. Use unions of relations to express that. For instance,  $< \cup =$  expresses  $\leq$ .
  - ▶  $2^3$  different relations (including the **impossible** and the **universal** relation)
  - ▶ Use **sets of atomic formulae** with these relations to describe **configurations**. For example:

$$\{x = y, y (< \cup >) z\}$$

- ▶ **Semantics:** Interpret the time point symbols and relation symbols over the **real** (or rational) numbers.

## Some Reasoning Problems

$$\{x (< \cup =) y, y (< \cup =) z, v (< \cup =) y, w > y, z (< \cup =) x\}$$

- ▶ **Satisfiability:** Are there values for all time points such that all formulae are satisfied?
- ▶ **Satisfiability** with  $v = w$ ?
- ▶ Finding a satisfying **instantiation** of all time points
- ▶ **Deduction:** Does  $x \{=\} y$  follow logically?  
Does  $v \leq w$  follow?
- ▶ Finding a **minimal description:** What are the most constrained relations that describe the same set of instantiations?

## From a Logical Point of View ...

In general, qualitatively described configurations are simple logical theories:

- ▶ Only sets of atomic formulae to describe the configuration
- ▶ Only existentially quantified variables (or constants)
- ▶ A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- ▶ We are interested in **satisfiability**, **model finding**, and **deduction**

Let  $\mathcal{B}$  be a finite set of (binary) relations on some (infinite) domain  $D$  (elements of  $\mathcal{B}$  are called **base relations**).

We require:

- ▶ The relations in  $\mathcal{B}$  are **JEPD**, i.e., jointly exhaustive and pairwise disjoint.
- ▶  $\mathcal{B}$  is **closed under converses**.

Then:

- ▶ Let  $\mathcal{A}$  be the set of relations that can be built by taking the unions of relations from  $\mathcal{B}$  ( $\rightsquigarrow 2^{|\mathcal{B}|}$  different relations).
- ▶  $\mathcal{A}$  is closed under converse, complement, intersection and union.
- ▶ Often,  $\mathcal{A}$  is **closed under composition of base relations**, i.e., for all  $B, B' \in \mathcal{B}$ ,

$$B \circ B' \in \mathcal{A}.$$

Then,  $\mathcal{A}$  is closed under composition of arbitrary relations.

But often this condition is not satisfied.

## Computing Operations on Relations

Let  $\mathcal{A}$  be the system of relations over a set of **base relations**  $\mathcal{B}$  that satisfies all the conditions above.

We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \cong \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

**Composition:**

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \circ B'_j$$

**Converse:**

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

**Complement:**

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathcal{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

**Intersection** and **union** are defined in the usual set-theoretical way.

## Reasoning Problems

Given a qualitative CSP:

**CSP-Satisfiability (CSAT):**

- ▶ Is the CSP satisfiable/solvable?

**CSP-Entailment (CENT):**

- ▶ Given in addition  $xRy$ : Is  $xRy$  satisfied in each solution of the CSP?

**Computation of an equivalent minimal CSPs (CMIN):**

- ▶ Compute for each pair  $x, y$  of variables the strongest constrained (minimal) relation entailed by the CSP.

## Reductions between CSP Problems

**Theorem**

*CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.*

**Proof.**

$\text{CSAT} \leq_T \text{CENT}$  and  $\text{CENT} \leq_T \text{CMIN}$  are obvious.

$\text{CENT} \leq_T \text{CSAT}$ : We solve CENT ( $\text{CSP} \models xRy?$ ) by testing satisfiability of the CSP extended by  $x\{B\}y$  where  $B$  ranges over all base relations. Let  $B_1, \dots, B_k$  be the relations for which we get a positive answer. Then  $x\{B_1, \dots, B_k\}y$  is entailed by the CSP.

$\text{CMIN} \leq_T \text{CENT}$ : We use entailment for computing the minimal constraint for each pair of variables. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed.  $\square$

## The Path Consistency Method

Given a qualitative CSP with  $R_{v_1, v_2} = R_{v_2, v_1}^{-1}$ . Then the **path consistency method** is to apply the operation

$$R_{v_1, v_2} \leftarrow R_{v_1, v_2} \cap (R_{v_1, v_3} \circ R_{v_3, v_2}).$$

on all the constraints of the network until a fixpoint is reached.

The path consistency method **guarantees** ...

- ▶ sometimes **minimality**
- ▶ sometimes **satisfiability**
- ▶ however sometimes the CSP is **not satisfiable**, even if the CSP contains only **base relations**

## Example: Point Relations

Composition table:

	<	=	>
<	<	<	<, =, >
=	<	=	>
>	<, =, >	>	>

Figure: Composition table for the point algebra. For example:  $\{<\} \circ \{=\} = \{<\}$

- ▶  $\{<, =\} \circ \{<\} = \{<\}$
- ▶  $\{<, >\} \circ \{<\} = \{<, =, >\}$
- ▶  $\{<, =\}^{-1} = \{>, =\}$
- ▶  $\{<, =\} \cap \{>, =\} = \{=\}$

## Some Properties of the Point Relations

### Theorem

A path consistent CSP over the point relations is satisfiable.

In particular, the path consistency method decides satisfiability.

### Theorem

A path consistent CSP over all point relations without  $\{<, >\}$  is minimal.

Proofs later ...

## A Pathological Relation System

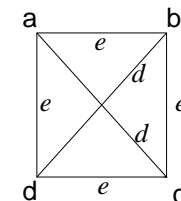
Let  $e, d, i$  be (self-converse) base relations between points on a circle:

- ▶  $e$ : Rotation by 72 degrees (left or right)
- ▶  $d$ : Rotation by 144 degrees (left or right)
- ▶  $i$ : Identity

Composition table:

$$\begin{aligned} e \circ e &= \{i, d\} \\ d \circ d &= \{i, e\} \\ e \circ d &= \{e, d\} \\ d \circ e &= \{e, d\} \end{aligned}$$

The following CSP is path-consistent and contains only base relations, but it is not satisfiable:



## Qualitative Constraint Languages

From now on, let  $D$  be a finite or infinite domain.

### Definition

A **partition scheme** on  $D$  is any non-empty, finite set  $\Delta$  of binary relations on  $D$  such that:

- ▶  $\Delta$  defines a partition of  $D \times D$ .
- ▶  $\Delta$  contains the binary identity relation  $\text{id}_D$ .
- ▶  $\Delta$  is closed under converses.

### Definition

A constraint language of binary relations on  $D$ ,  $\Gamma$ , is said to be **generated** from a partition scheme  $\Delta$ , if  $\Gamma$  consists of all finite unions of relations in  $\Delta$ .

Constraint languages in this sense will be referred to as **qualitative constraint languages**.

## Qualitative Constraint Network

Let  $\Gamma$  be a subset of a qualitative constraint language with partition scheme  $\Delta$ .

### Definition

A **qualitative constraint network** over  $\Gamma$  is a triple

$$P = \langle V, D, C \rangle,$$

where:

- ▶  $V$  is a non-empty and finite set of **variables**,
- ▶  $D$  is an arbitrary non-empty set (**domain**),
- ▶  $C$  is a finite set of **constraints**  $C_1, \dots, C_q$ , i.e., each constraint  $C_i$  is a pair  $(s_i, R_i)$ , where  $s_i$  is a pair of variables and  $R_i$  is a binary relation contained in  $\Gamma$ .

## Weak Composition

Let  $\Gamma$  be a qualitative constraint language with partition scheme  $\Delta$ . For  $R, S \in \Gamma$ , define:

$$R \circ_w S := \bigcup \{ T \in \Delta : T \cap (R \circ S) \neq \emptyset \}$$

$\circ_w$  is called **weak composition** of  $R$  and  $S$ .

### Lemma

For all relations  $R, S, T \in \Gamma$ ,

- ▶  $R \circ S \subseteq R \circ_w S$ ;
- ▶  $T \cap (R \circ S) = \emptyset$  if and only if  $T \cap (R \circ_w S) = \emptyset$ ;
- ▶  $(R \circ_w S)^{-1} = S^{-1} \circ_w R^{-1}$ ;
- ▶  $R \circ_w (S \cup T) = (R \circ_w S) \cup (R \circ_w T)$ .

## Weak Composition: Examples

### Example:

Consider a linear order on a domain with 2 elements  $a < b$ . The relations  $R_<, R_=:, R_>$  define a partition schema on  $D$ . It holds:

$$R_< \circ R_< = R_> \circ R_> = \emptyset, \quad R_< \circ R_> = \{(a, a)\}, \quad R_> \circ R_< = \{(b, b)\}$$

but

$$R_< \circ_w R_< = R_> \circ_w R_> = \emptyset, \quad R_< \circ_w R_> = R_=:, \quad R_> \circ_w R_< = R_=:$$

Moreover,

$$(R_< \circ_w R_>) \circ_w R_> = R_=: \circ_w R_> = R_> \neq \emptyset = R_< \circ_w \emptyset = R_< \circ_w (R_> \circ_w R_>).$$

### Example:

Consider a linear order on a domain with 3 elements  $a < b < c$ . Then

$$R_< \circ R_< = \{(a, c)\} \quad \text{but} \quad R_< \circ_w R_< = R_<.$$

## Non-Associative Relation Algebras

### Definition

A **non-associative relation algebra** is a set  $A$  with

- ▶ binary operations  $\sqcap$ ,  $\sqcup$ , and  $;$ ,
- ▶ unary operations  $-$  and  $^{\neg}$ , and
- ▶ distinct elements  $0$ ,  $1$ , and  $\delta$  such that

(a)  $(A, \sqcap, \sqcup, -, 0, 1)$  is a Boolean algebra.

(b) For all elements  $a$ ,  $b$  and  $c$  of  $A$ :

$$a ; (b \sqcup c) = (a ; b) \sqcup (a ; c)$$

$$\delta ; a = a ; \delta = a$$

$$(a^{\neg})^{\neg} = a \text{ and } (-a)^{\neg} = -(a^{\neg})$$

$$(a \sqcup b)^{\neg} = a^{\neg} \sqcup b^{\neg}$$

$$(a ; b)^{\neg} = b^{\neg} ; a^{\neg}$$

$$(a ; b) \sqcap c^{\neg} = 0 \text{ if and only if } (b ; c) \sqcap a^{\neg} = 0$$

## Qualitative Languages and Algebras

Let  $\Gamma$  be a qualitative constraint language with partition scheme  $\Delta$ . As spelled out before, each relation  $R$  in  $\Gamma$  can be represented by a finite disjunction of “base relations”  $B_1, \dots, B_k \in \Delta$ . In what follows we identify  $R$  with the set of its base relations

$$\{B_1, \dots, B_k\}.$$

### Lemma

For each partition scheme  $\Delta$ , the tuple

$$\langle 2^{\Delta}, \cap, \cup, \circ_w, \mathbf{C}_{\Delta}, ^{-1}, \emptyset, \Delta, \text{id}_{\Delta} \rangle$$

defines a non-associative relation algebra.

## Algebraically Closed Networks

A qualitative network  $P = \langle V, D, C \rangle$  is **normalized**, if

- ▶ for each pair of variables  $x, y$ ,  $C$  contains at least one constraint  $((x, y), R)$ ;
- ▶ for each constraint  $((x, x), R)$  in  $C$ ,  $R = \text{id}_D$ ;
- ▶ for constraints  $((x, y), R)$  and  $((y, x), S)$  in  $C$ ,  $R = S^{-1}$ .

In what follows we will always assume that constraint networks are normalized.

### Definition

A qualitative constraint network  $P$  is **algebraically closed** (or: **a-closed**), if for all constraints  $((x, y), R)$ ,  $((x, z), S)$ , and  $((z, y), T)$  of  $P$ , it holds:

$$R \subseteq S \circ_w T.$$

Note: If  $P$  is algebraically closed, then  $R = R \cap (S \circ_w T)$ .

## Constraint Propagation

The **path consistency algorithm** can only be used if the underlying partition scheme is closed under composition, i.e., if for each pair of relations  $R, S \in \Delta$ ,  $R \circ S$  is a (finite) union of a subset of  $\Delta$ .

The **algebraic closure algorithm** is a variant of the path consistency algorithm. Instead of ordinary composition of relations, we use weak composition.

Since weak composition is an upper approximation of composition only, the algebraic closure algorithm may not result in a path-consistent network.

Let  $P = \langle V, D, C \rangle$  be a (normalized) qualitative constraint network. Let  $\text{Table}[i, j]$  be a  $n \times n$ -matrix ( $n$ : number of variables), in which we record the constraints between the variables.

## Algebraic Closure Algorithm

### EnforceAlgClosure ( $P$ ):

*Input:* a qualitative network  $P = \langle V, D, C \rangle$

*Output:* “inconsistent”, or an equivalent algebraically closed network  $P'$

$$\text{Paths}(i, j) = \{(i, j, k) : 1 \leq k \leq n, k \neq i, j\} \cup \{(k, i, j) : 1 \leq k \leq n, k \neq i, j\}$$

$$\text{Queue} := \bigcup_{i, j} \text{Paths}(i, j)$$

**while**  $Q \neq \emptyset$

  select and delete  $(i, k, j)$  from  $Q$

$$T := \text{Table}[i, j] \cap (\text{Table}[i, k] \circ_w \text{Table}[k, j])$$

**if**  $T = \emptyset$

**return** “inconsistent”

**elseif**  $T \neq \text{Table}[i, j]$

$$\text{Table}[i, j] := T$$

$$\text{Table}[j, i] := T^{-1}$$

$$\text{Queue} := \text{Queue} \cup \text{Paths}(i, j)$$

**return**  $P'$  with the refined constraints as recorded in  $\text{Table}$

## Computing on the Symbolic Level

Let  $\Gamma$  be a qualitative constraint language with partition scheme  $\Delta$ . We suppose that we have determined (by some formal proof or some computation) the (weak) composition table for  $\Delta$ , i.e.,

$$\circ_{(w)} : \Delta \times \Delta \rightarrow 2^\Delta.$$

Let now  $B$  be a finite set of symbols (bijective with  $\Delta$ ).

Then  $2^B$  is a Boolean algebra, from which we obtain a (non-associative) relation algebra, if we extend  $\circ_{(w)}$  to a function

$$\circ_{(w)} : 2^B \times 2^B \rightarrow 2^B.$$

Now we can perform all the operations needed in the path consistency/a-closure algorithm on the symbolic level.

## Path Consistency and Tractability

Let  $\Gamma$  be a subset of a qualitative constraint language with a partition scheme  $\Delta$  that is closed under composition.

Let  $\hat{\Gamma}$  be smallest superset of  $\Gamma$  that is closed under intersection, converses, and composition.

### Lemma

*There exists a polynomial time reduction from  $\text{CSP}(\hat{\Gamma})$  to  $\text{CSP}(\Gamma)$ , provided  $\Gamma$  contains identity and the universal relation. In particular, it holds:*

- ▶  $\Gamma$  is tractable if and only if  $\hat{\Gamma}$  is tractable.
- ▶ Enforcing path consistency decides satisfiability over  $\hat{\Gamma}$  if and only if it does so over  $\Gamma$ .

### Proof idea.

Each relation in  $\hat{\Gamma}$  stems from a finite number of compositions, intersections, and conversions applied to relations in  $\Gamma$ . Hence each constraint network over  $\hat{\Gamma}$  can be transformed step-by-step into an equivalent network over  $\Gamma$ . In the case where a relation results from composing other relations, we need to introduce some fresh variables.  $\square$

## Algebraic Closure and Tractability

Let now  $\Gamma$  be a subset of a qualitative constraint language with a partition scheme  $\Delta$  (not necessarily closed under composition).

Let  $\hat{\Gamma}^w$  be smallest superset of  $\Gamma$  that is closed under intersection, converses, and weak composition.

### Lemma (Ligozat & Renz 2005)

*If enforcing a-closure decides satisfiability for atomic networks (i.e., for qualitative networks over  $\Delta$ ), then  $\text{CSP}(\hat{\Gamma}^w)$  is polynomial-time reducible to  $\text{CSP}(\Gamma)$  (provided  $\Gamma$  contains identity and the universal relation. In particular, if a-closure decides satisfiability for atomic networks, then*

- ▶  $\Gamma$  is tractable if and only if  $\hat{\Gamma}^w$  is so;
- ▶ enforcing a-closure decides satisfiability over  $\Gamma$  if and only if a-closure decides satisfiability over  $\hat{\Gamma}^w$ .

## Allen's Interval Calculus

- ▶ Allen's interval calculus (IA): **time intervals** and **binary relations** over them
- ▶ Let  $\langle \mathbb{R}, < \rangle$  be the linear order on the real numbers (conceived of as the **flow of time**).  
Then, the **domain**  $D$  of Allen's calculus is the set of all *intervals*

$$X = (X^-, X^+) \in \mathbb{R}^2, \text{ where } X^- < X^+$$

(naïve approach)

- ▶ **Relations** between concrete intervals, e. g.:  
 $(1.0, 2.0)$  strictly before  $(3.0, 5.5)$   
 $(1.0, 3.0)$  meets  $(3.0, 5.5)$   
 $(1.0, 4.0)$  overlaps  $(3.0, 5.5)$   
 ...

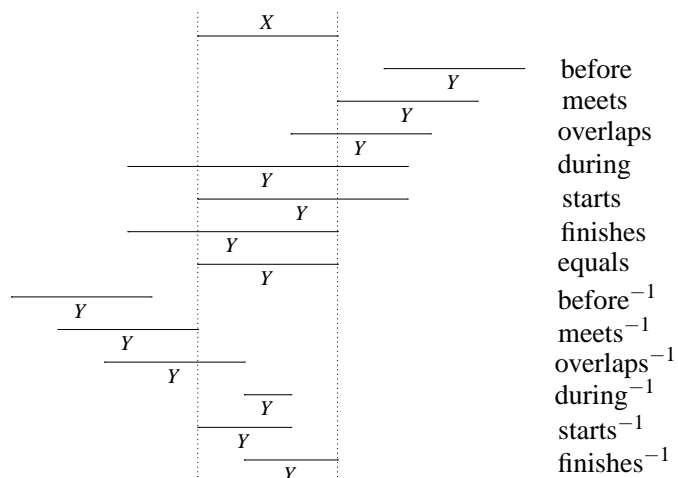
## IA: The Base Relations

To determine all possible relation between Allen intervals, we determine how one can order the four points of two intervals:

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	$\mathfrak{m}$	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	$\circ$	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	$\mathfrak{s}$	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	$\mathfrak{f}$	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	$\mathfrak{d}$	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the *converse* relations (obtained by exchanging  $X$  and  $Y$ )

## IA: The 13 Base Relations Graphically



## IA: Partition Scheme and Composition

### Lemma

The 13 base relations of Allen's interval calculus define a *partition scheme* on the set of all Allen intervals.

In what follows:

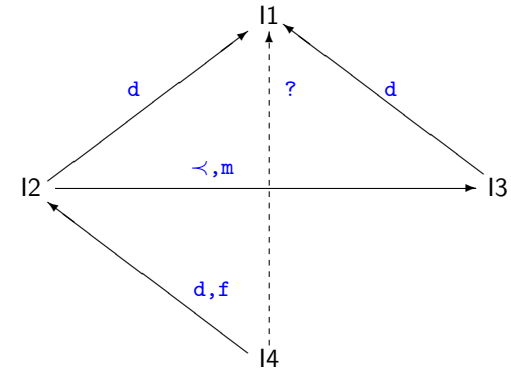
- ▶ **IA**: the qualitative constraint language generated from all base relations of Allen's interval calculus (contains  $2^{13} = 8192$  relations)
- ▶ **IA-B**: the subclass of IA containing base relations only

### Lemma

The set of base relations of Allen's interval calculus is closed under composition.

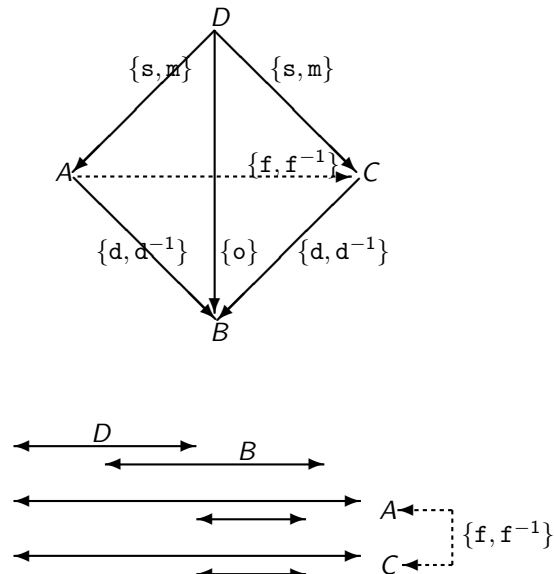


### IA: An Example



Compose the constraints:  $I4 \{d, f\} I2$  and  $I2 \{d\} I1$ :  $I4 \{d\} I1$ .

### IA: Example for Incompleteness



### IA: NP-Hardness

#### Theorem (Kautz & Vilain)

Deciding satisfiability over IA is NP-hard.

#### Proof.

Reduction from 3-colorability (the original proof uses 3Sat).

Let  $G = (V, E)$ ,  $V = \{v_1, \dots, v_n\}$  be an instance of 3-colorability.

Then we use the intervals  $\{v_1, \dots, v_n, 1, 2, 3\}$  with the following constraints:

$$\begin{array}{lcl} 1 & \{m\} & 2 \\ 2 & \{m\} & 3 \\ v_i & \{m, \equiv, m^{-1}\} & 2 \quad \forall v_i \in V \\ v_i & \{m, m^{-1}, <, >\} & v_j \quad \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable iff  $G$  can be colored with 3 colors. □

## IA: Clause Representation

Following, we will look at polynomial special cases, i.e., subclasses of the qualitative constraint language IA.

For this we start from a natural translation of interval relations/constraints (of the form  $X R Y$ ) into **clause formulas** over **atoms** of the form  $a op b$ , where:

- ▶  $a, b \in \{X^-, X^+, Y^-, Y^+\}$ ; and
- ▶  $op \in \{<, >, =, \leq, \geq\}$ .

**Example:** All base relations can be expressed as unit clauses.

### Lemma

Let  $P$  be a constraint network over IA, and let  $\pi(P)$  be the translation of  $P$  into clause form.

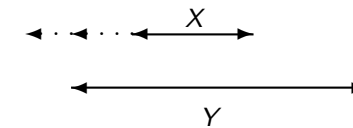
$P$  is satisfiable iff  $\pi(P)$  is satisfiable over the real numbers.

## IA: The Continuous Endpoint Class

**Continuous Endpoint Class IA-C:** the subset of IA consisting of those relations with a clause form containing only unit clauses, where  $\neg(a = b)$  is **forbidden**.

**Example:** All basic relations and, e.g.,  $\{d, o, s\}$ , because

$$\pi(X \{d, o, s\} Y) = \begin{cases} X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+ \end{cases}$$



The set IA-C contains 83 relations. It is **closed** under intersection, composition, and converses (it is a **sub-algebra** wrt. these three operations on relations). This can be shown by using a computer program.

## IA: Consistency for IA-C

Following we prove:

### Lemma

Each 3-consistent interval CSP over IA-C is globally consistent.

From this we can conclude:

### Theorem (van Beek)

Applied to networks over IA-C, enforcing path consistency decides satisfiability and solves the minimal label problem.

### Corollary

A path-consistent interval constraint network containing base relations only is satisfiable.

## Helly's Theorem

### Definition

A set  $M \subseteq \mathbb{R}^n$  is **convex** iff for all pairs of points  $a, b \in M$ , all points on the line connecting  $a$  and  $b$  belong to  $M$ .

### Theorem (Helly)

Let  $F$  be a family of at least  $n + 1$  convex sets in  $\mathbb{R}^n$ . If all sub-families of  $F$  with  $n + 1$  sets have a non-empty intersection, then  $\bigcap F \neq \emptyset$ .

IA: Strong  $n$ -Consistency (1)

## Proof of the lemma.

We prove the claim by induction over  $k$  with  $k \leq n$ .

**Base case:**  $k = 1, 2, 3$  ✓

**Induction assumption:** Assume strong  $k - 1$ -consistency (and non-emptiness of all relations)

**Induction step:** From the assumption, it follows that there is an instantiation of  $k - 1$  variables  $X_i$  to pairs  $(s_i, e_i) \in \mathbb{R}^2$  satisfying the constraints  $R_{ij}$  between the  $k - 1$  variables.

We have to show that we can extend the instantiation to any  $k$ th variable.

IA: Strong  $n$ -Consistency (2): Instantiating the  $k$ th Variable

## Proof (Part 2).

The instantiation of the  $k - 1$  variables  $X_i$  to  $(s_i, e_i)$  restricts the instantiation of  $X_k$ .

**Note:** Since  $R_{ij} \in \text{IA-C}$  by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in  $\mathbb{R}^2$ .

↪ Consider sets of 3 inequalities (= 3 convex sets).

IA: Strong  $n$ -Consistency (3): Using Helly's Theorem

## Proof (Part 3).

**Case 1:** All 3 inequalities mention only  $X_k^-$  (or mention only  $X_k^+$ ). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

**Case 2:** The inequalities mention  $X_k^-$  and  $X_k^+$ , but it does not contain the inequality  $X_k^- < X_k^+$ . Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

**Case 3:** The set contains the inequality  $X_k^- < X_k^+$ . In this case, only three intervals (incl.  $X_k$ ) can be involved and by the same argument as above there exists a common point.

↪ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.

↪ Strong  $k$ -consistency for all  $k \leq n$ .

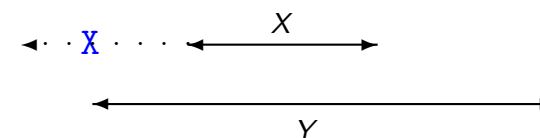
□

## IA: The Endpoint Subclass

**Endpoint Subclass:** IA- $\mathcal{P}$  is the subclass that permits a clause form containing only **unit** clauses ( $a \neq b$  is now allowed).

**Example:** all basic relations and  $\{d, o\}$  since

$$\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+\}$$



**Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)**

*The path consistency method decides satisfiability over IA- $\mathcal{P}$ .*

## IA: The ORD-Horn Subclass

**ORD-Horn Subclass:**  $IA-\mathcal{H}$  is the subclass of IA that permits a clause form containing only **Horn clauses**, where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$  is not allowed!

**Example:** all  $R \in IA-\mathcal{P}$  and  $\{o, s, f^{-1}\}$ :

$$\pi(X\{o, s, f^{-1}\}Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

## Partial Orders: The ORD Theory

Let  $ORD$  be the following theory:

$$\begin{array}{ll} \forall x, y, z: & x \leq y \wedge y \leq z \rightarrow x \leq z \quad (\text{transitivity}) \\ \forall x: & x \leq x \quad (\text{reflexivity}) \\ \forall x, y: & x \leq y \wedge y \leq x \rightarrow x = y \quad (\text{anti-symmetry}) \\ \forall x, y: & x = y \rightarrow x \leq y \quad (\text{weakening of } =) \\ \forall x, y: & x = y \rightarrow y \leq x \quad (\text{weakening of } =). \end{array}$$

- ▶  $ORD$  describes partially ordered sets,  $\leq$  being the ordering relation.
- ▶  $ORD$  is a **Horn theory**
- ▶ What is missing wrt. *dense* and *linear* orders?

## Satisfiability over Partial Orders

### Lemma

Let  $\Theta$  be a CSP over  $IA-\mathcal{H}$ .  $\Theta$  is satisfiable over interval interpretations iff  $\pi(\Theta) \cup ORD$  is satisfiable over arbitrary interpretations.

### Proof.

$\Rightarrow$ : Since the reals form a partially ordered set (i.e., satisfy  $ORD$ ), this direction is trivial.

$\Leftarrow$ : Each extension of a partial order to a linear order satisfies all formulae of the form  $a \leq b$ ,  $a = b$ , and  $a \neq b$  which have been satisfied over the original partial order.  $\square$

## Complexity of CSAT( $IA-\mathcal{H}$ )

Let  $ORD_{\pi(\Theta)}$  be the propositional theory resulting from instantiating all axioms with the endpoints occurring in  $\pi(\Theta)$ .

### Lemma

$ORD \cup \pi(\Theta)$  is satisfiable iff  $ORD_{\pi(\Theta)} \cup \pi(\Theta)$  is so.

**Proof idea:** Herbrand expansion!

### Theorem

$CSAT(IA-\mathcal{H})$  can be decided in polynomial time.

### Proof.

$CSAT(IA-\mathcal{H})$  instances can be translated into a propositional Horn theory with blowup  $O(n^3)$  according to the previous Prop., and such a theory is decidable in polynomial time.  $\square$

$IA-\mathcal{C} \subset IA-\mathcal{P} \subset IA-\mathcal{H}$  with  $|IA-\mathcal{C}| = 83$ ,  $|IA-\mathcal{P}| = 188$ ,  $|IA-\mathcal{H}| = 868$

## Path Consistency and the OH-Class

### Lemma

Let  $\Theta$  be a path-consistent set over  $IA-\mathcal{H}$ . Then

$$(X\{Y\}) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

Proof idea: One can show that  $ORD_{\pi(\Theta)} \cup \pi(\Theta)$  is closed wrt. **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows.

### Theorem

Enforcing path consistency decides  $CSAT(IA-\mathcal{H})$ .

- ↔ Maximality of  $IA-\mathcal{H}$  ?
- ↔ Do we have to check all 8192 – 868 extensions?

## IA: The ORD-Horn Subclass: Maximality

A computer-aided case analysis leads to the following result:

### Lemma

There are only two minimal sub-algebras containing all base relations that strictly contain  $IA-\mathcal{H}$ :  $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause forms of these relations contain “proper” disjunctions!

### Theorem

The satisfiability problem over  $IA-\mathcal{H} \cup \{N_i\}$  is NP-complete.

### Lemma

$IA-\mathcal{H}$  is the **only** maximal tractable subclass that contains all base relations of  $IA$ .

## IA: Solving General Allen CSPs

- ▶ Backtracking algorithm using **path consistency** as a **forward-checking method**
- ▶ Method works on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- ▶ Refinements and evaluation of different heuristics
- ↔ Which tractable fragment should one use?

## IA: Branching Factors

- ▶ If the labels are split into **base relations**, then on average a label is split into  
**6.5 relations**
- ▶ If the labels are split into **pointizable relations** ( $\mathcal{P}$ ), then on average a label is split into  
**2.955 relations**
- ▶ If the labels are split into **ORD-Horn relations** ( $\mathcal{H}$ ), then on average a label is split into  
**2.533 relations**
- ↔ A difference of **0.422** which becomes significant, when applied to extremely hard instances

## RCC8: Motivation

We may want to state qualitative relationships between regions in space, for example:

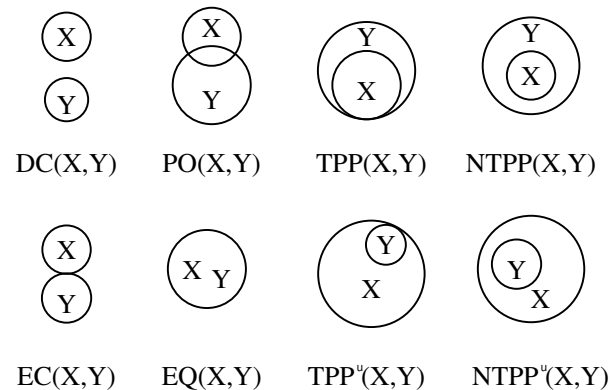
- ▶ “Region  $X$  touches region  $Y$ ”
- ▶ “Germany and Switzerland have a common border”
- ▶ “Freiburg is located in Baden-Württemberg”

## RCC8: Possible Applications

- ▶ This can be useful when only **partial information** is available:
  - ▶ We may know that region  $X$  is **not connected** with region  $Y$  without knowing the shape and location of  $X$  and  $Y$ .
- ▶ We may want to **query** a database:
  - ▶ Show me all countries **bordering** the Mediterranean!
- ▶ We may want to state **integrity constraints**:
  - ▶ An island has to be located **in the interior of** a sea.

## RCC8: Qualitative Relations Between Regions

Eight relations between **regions**:



## RCC8: Intuition

- ▶ **Regions** are some “reasonable” non-empty subsets of space.
- ▶ **DC** (disconnected) means that the two regions do not share any point at all.
- ▶ **EC** (externally connected) means that they only share **borders**.
- ▶ **PO** (partially overlapping) means that the two regions share **interior** points.
- ▶ **TPP** (tangential proper part) means that one region is a subset of the other sharing some points on the **borders**.
- ▶ **NTPP** (non-tangential proper part) same, but without sharing any bordering points.

## Point-Set Topology

**Point-set topology** is a mathematical theory that deals with properties of space independent of size and shape.

In topology, we can define notions such as

- ▶ **interior** and **exterior** points of regions,
- ▶ **isolated points** of regions,
- ▶ **boundaries** of regions,
- ▶ **connected components** of regions,
- ▶ **connected regions**,
- ▶ ...

## Topology

### Definition

A **topological space** is a pair  $\mathcal{T} = (S, \mathcal{O})$ , where

- ▶  $S$  is a non-empty set (the **universe**), and
- ▶  $\mathcal{O}$  is a set of subsets of  $S$  (the **open sets**)

such that the following conditions hold:

- ▶  $\emptyset \in \mathcal{O}$  and  $S \in \mathcal{O}$ .
- ▶ If  $O_1 \in \mathcal{O}$  and  $O_2 \in \mathcal{O}$ , then  $O_1 \cap O_2 \in \mathcal{O}$ .
- ▶ If  $(O_i)_{i \in I}$  is a (possibly infinite) family of elements from  $\mathcal{O}$ , then

$$\bigcup_{i \in I} O_i \in \mathcal{O}.$$

**Example:** In Euclidean space, a set  $O$  is open if for each point  $x \in O$  there is a ball surrounding  $x$  that is contained in  $O$ .

## Terminology & Notation

### Definition

Let  $X \subseteq S$  and  $x \in S$ .

- ▶ A set  $N \subseteq S$  is a **neighborhood** of a point  $x$  if there is an open set  $O \in \mathcal{O}$  such that  $x \in O \subseteq N$ .
- ▶  $x \in S$  is an **interior point** of  $X$  if there is a neighborhood  $N$  of  $x$  such that  $N \subseteq X$ .
- ▶  $x \in S$  is a **touching point** of  $X$  if every neighborhood of  $x$  has a non-empty intersection with  $X$ .

### Notation:

- ▶  $\text{int}(X)$  is the set of **interior points** of  $X$  (the **interior** of  $X$ ).
- ▶  $\text{cls}(X)$  is the set of **touching points** of  $X$  (the **closure** of  $X$ ).
- ▶ A set is **closed** if  $X = \text{cls}(X)$ .

## Interior and Closure Operators

The function  $\text{int}(\cdot)$  is an **interior operator**:

1.  $\text{int}(S) = S$
2.  $\text{int}(X) \cap \text{int}(Y) = \text{int}(X \cap Y)$
3.  $\text{int}(X) \subseteq X$
4.  $\text{int}(\text{int}(X)) = \text{int}(X)$

### Note:

- ▶  $X$  is **open** iff  $X = \text{int}(X)$
- ▶  $\text{cls}(X) = S \setminus \text{int}(S \setminus X)$

## RCC8: What Is a Region?



A and D are **reasonable** regions, B, C, and E are not

In other words,  $X$  is a **region** iff it is **non-empty**

$$X \neq \emptyset$$

and **regular closed**, i. e., the closure of an open set:

$$X = \text{cls}(\text{int}(X)).$$

It is not necessary that a region is **internally connected**.

## Defining the RCC8-Relations

Let  $S$  be a topological space. Then define the following relations on  $\text{Reg}$ :

$$\text{DC}(X, Y) := X \cap Y = \emptyset$$

$$\text{EC}(X, Y) := X \cap Y \neq \emptyset \wedge \text{int } X \cap \text{int } Y = \emptyset$$

$$\text{PO}(X, Y) := \text{int } X \cap \text{int } Y \neq \emptyset \wedge X \not\subseteq Y \wedge Y \not\subseteq X$$

$$\text{EQ}(X, Y) := X = Y$$

$$\text{TPP}(X, Y) := X \subseteq Y \wedge X \not\subseteq \text{int } Y$$

$$\text{NTPP}(X, Y) := X \subseteq \text{int } Y$$

↪ It can be seen that these relations define a partition scheme.

## RCC8: From Regions to Boolean Algebras

Let  $\text{Reg}$  denote the set of all regular closed set of some fixed topological space.

For  $X, Y \in \text{Reg} \cup \{\emptyset\}$  define:

$$\neg X := \text{cls}(S \setminus X)$$

$$X \sqcup Y := X \cup Y$$

$$X \sqcap Y := \text{cls}(\text{int}(X \cap Y))$$

By these definition, we obtain a Boolean algebra.

## Boolean Connection Algebras

## Definition

A **connection algebra** is a Boolean algebra  $B$  together with a binary relation  $C$  on  $B$  such that the following conditions are satisfied:

- ▶  $x \neq 0 \Leftrightarrow x C x$
- ▶  $x C y \Rightarrow y C x$
- ▶  $x \neq 0, 1 \Rightarrow x C \neg x$
- ▶  $x C y \cup z \Leftrightarrow x C y \text{ or } x C z$
- ▶  $x \neq 0, 1 \Rightarrow \text{not } x C y, \text{ for some } y \neq 0, 1$



## RCC8: From Topologies to Connection Algebras

If the underlying topological space is regular and connected, i.e.,

- ▶ Hausdorff and for each  $x \in S$  and closed subset  $A \subseteq S$  with  $x \notin A$ , there exist disjoint open neighborhoods of  $x$  and  $A$ ;
- ▶ the only sets that are open and closed are  $\emptyset$  and  $S$ ;

then

$$x C y \iff x \cap y \neq \emptyset$$

defines a connection algebra on  $\text{Reg} \cup \{\emptyset\}$ .

## Defining the RCC8-Relations (2)

Let  $B$  be a connection algebra. Then we can define the RCC8 relations on  $B \setminus \{0\}$  as follows:

$$X DC Y := \text{not } X C Y$$

$$X P Y := (X, Y) \notin C \circ DC$$

$$X PP Y := X P Y \wedge X \neq Y$$

$$X O Y := (X, Y) \in P^{-1} \circ P$$

$$X PO Y := X O Y \wedge \text{not } X P Y \wedge \text{not } Y P X$$

$$X EC Y := X C Y \wedge \text{not } X O Y$$

$$X TPP Y := X PP Y \wedge (X, Y) \in EC \circ EC$$

$$X NTPP Y := X PP Y \wedge \text{not } X TPP Y$$

...

## RCC8: Complexity

Using a reduction from 3SAT, it can be shown:

### Theorem

*Testing satisfiability over arbitrary RCC8 relations is NP-hard.*

Using a translation into S4-modal logics, one can show:

### Theorem

*Testing satisfiability over arbitrary RCC8 relations is NP-complete.*

## Lower Bound: Proving NP-Hardness

- ▶ **Idea:** Reduction from 3-SAT

- ▶ **3-SAT** structure

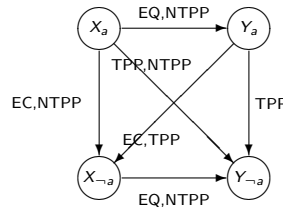
1. Literals  $a, b, c$ : can be true or false
2. Complementary literals:  $a$  is true iff  $\neg a$  is false
3. Clauses  $l_1 \vee l_2 \vee l_3$ : at least one literal must be true

- ▶ **RCC8-CSP**

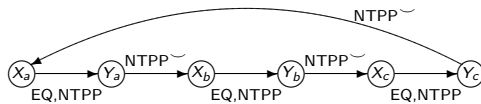
1. **Truth value constraints**  $X_a\{R_t, R_f\}Y_a$ : Either  $X_a\{R_t\}Y_a$  or  $X_a\{R_f\}Y_a$  holds
2. **Polarity constraints**:  $X_a\{R_t\}Y_a$  holds iff  $X_{\neg a}\{R_f\}Y_{\neg a}$  holds
3. **Clause constraints**: At least one of  $X_{l_1}\{R_t\}Y_{l_1}$ ,  $X_{l_2}\{R_t\}Y_{l_2}$ , or  $X_{l_3}\{R_t\}Y_{l_3}$  holds

### The Reduction

- Relations:  $R_t = NTPP, R_f = EQ$
- Polarity constraints:



- Clause constraints:



- RCC8 sat.  $\Rightarrow$  3-SAT: follows from reduction
- 3-SAT  $\Rightarrow$  RCC8 sat.: Construction of model for  $\Theta_\phi$  for each positive 3-SAT instance  $\phi$

### RCC8: Constraint Propagation

- As in Allen's interval algebra, we may want to use **constraint propagation** instead of translating everything to modal logic.
- We need a composition table ...
- ... which could be computed using the modal logic encoding (and in fact, this has been done).
- Based on this table, we can then apply the algebraic closure algorithm
- ... and ask ourselves for which fragment of RCC8 it is **complete**.

### RCC8: Composition Table

o	DC	EC	PO	TPP	NTPP	TPP <sup>-1</sup>	NTPP <sup>-1</sup>	EQ
DC	*	DC, EC PO, TPP NTPP	DC, EC PO, TPP NTPP	DC, EC PO, TPP NTPP	DC, EC PO, TPP NTPP	DC	DC	DC
EC	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	DC, EC PO, TPP TPP <sup>-1</sup> , EQ	DC, EC PO, TPP NTPP	EC, PO TPP NTPP	PO TPP NTPP	DC, EC	DC	EC
PO	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	*	PO TPP NTPP	PO TPP NTPP	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO
TPP	DC	DC, EC	DC, EC PO, TPP NTPP	TPP NTPP	NTPP	DC, EC PO, TPP TPP <sup>-1</sup> , EQ	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	TPP
NTPP	DC	DC	DC, EC PO, TPP NTPP	NTPP	NTPP	DC, EC PO, TPP NTPP	*	NTPP
TPP <sup>-1</sup>	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	EC, PO TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO, EQ TPP TPP <sup>-1</sup>	PO TPP NTPP	TPP <sup>-1</sup> NTPP <sup>-1</sup>	NTPP <sup>-1</sup>	TPP <sup>-1</sup>
NTPP <sup>-1</sup>	DC, EC PO, TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO TPP <sup>-1</sup> NTPP <sup>-1</sup>	PO, TPP <sup>-1</sup> TPP, NTPP NTPP <sup>-1</sup> , EQ	NTPP <sup>-1</sup>	NTPP <sup>-1</sup>	NTPP <sup>-1</sup>
EQ	DC	EC	PO	TPP	NTPP	TPP <sup>-1</sup>	NTPP <sup>-1</sup>	EQ

### RCC8: Is the Composition Table Extensional?

It can easily be verified that already in the 2-dimensional case, the set of base relations is not closed under composition:

- Consider  $EC \circ TPP$  and  $X NTPP S$ , where  $S$  denotes the universal region.
- Consider  $EC \circ EC$  and a donut-like region  $X$  with "hole"  $Y$ .

**Lemma (Düntsch et al. 2001)**

*In each connection algebra, the relation algebra generated by the RCC8 base relations contains at least 25 atomic relations.*

**Lemma (Li et al. 2006)**

*In each model associated to some Euclidean space  $\mathbb{R}^n$ , the relation algebra generated by the RCC8 base relations contains an infinite strictly decreasing sequence of relations.*

## RCC8: Tractable Fragments?

### Theorem (Li 2006)

Enforcing algebraic closure on atomic RCC8 constraint network decides satisfiability.

- ▶ As in the case of Allen's interval calculus, we may ask for **maximal tractable subsets** ...
- ▶ Again, one can identify relations that can be encoded by **Horn formulae** ...
- ▶ 148 **Horn relations**  $\mathcal{H}_8$ , which forms again a **maximal subset**.
- ▶ There are **2 additional maximal subsets** that allow for poly. satisfiability testing!

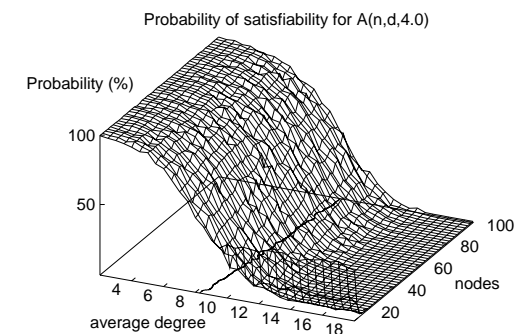
## Some Experiments

- ▶ How difficult is the RCC8 satisfiability problem **in practice**?
- ▶ Are there particularly **difficult instances**?
  - ↪ Where is the **phase transition** region?
  - ↪ Cheeseman et al [IJCAI 91] conjectured that for all NP-complete problems there exists a very small range – the **phase transition region** – where the probability of satisfiability of **randomly generated** instances changes from 1 to 0. They also conjectured that in this area one finds many **hard** instances.
- ▶ How well does the path consistency method **approximate** satisfiability?
- ▶ Can  $\mathcal{H}_8$  be used to speed up the satisfiability testing?

## Generating Instances

- ▶ Randomly generating instances according to the following **parameters**:
  - ▶ **Number of nodes**  $n$
  - ▶ Average **number of constraints**  $d$ : ( $nd/2$  out of  $n(n-1)/2$  possible constraints)
  - ▶ Average **number of base relations**  $l$  per constraint
  - ▶ Allowed constraints
    - ▶  $A(n, d, l)$ : all RCC8 relations
    - ▶  $H(n, d, l)$ : only relations out of  $RCC8 - \mathcal{H}_8$

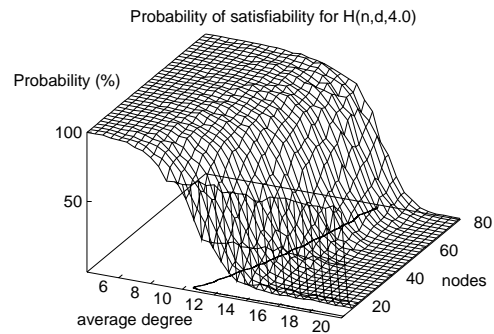
## Phase Transition for $A(n, d, 4)$



500 instances per data point

- ▶ **Phase transition** for  $A(n, d, 4)$  between  $d = 8$  and  $d = 10$  for  $10 \leq n \leq 100$ .

## Phase Transition for $H(n, d, 4)$

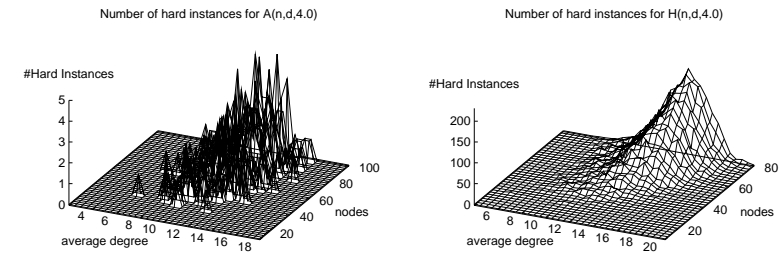


500 instances per data point

- Phase transition for  $H(n, d, 4)$  between  $d = 10$  and  $d = 15$  for  $10 \leq n \leq 80$ .

## Hard Instances ...

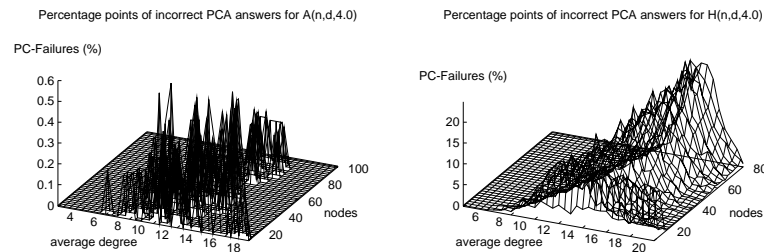
... using more than 10,000 search nodes



500 instances per data point

## Quality of Path Consistency...

... measured as the percentage of path consistent but unsatisfiable CSPs







500 instances per data point





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