

# Constraint Satisfaction Problems

## Constraint Optimization

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# Constraint Satisfaction Problems

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# Hard and Soft Constraint

Real-life problems often contain **hard** and **soft constraints**:

**Hard constraints**: must be satisfied;

**Soft constraints**: should be satisfied, but may be violated.

Example: In time-tabling problems,

- ▶ resource constraints such as “a teacher can teach only one class at a time” must be satisfied;
- ▶ a request such as “the schedule of teacher should be concentrated in two days” is simply a preference, but not essential for the solution.

What to do with soft constraints?

# Constraint Optimization

Formalizing problems with soft and hard constraints leads to constraint networks augmented with a **global cost function** (also called **criterion function** or **objective function**), based on the satisfaction of soft constraints.

A **constraint optimization problem** (COP) is the problem of finding a variable assignment to all variables that satisfies all hard constraints and at the same time optimizes the global cost function.

Note: Every constraint satisfaction problem can be viewed as a constraint optimization problem – when not all constraints are satisfiable. Try to find an assignment that maximizes the number of satisfied constraints: **MAX-CSP** problem.

## Example 1: Power Plant Maintenance

Given

1. a number of power generators,
2. preventive maintenance intervals,
3. time for maintenance,
4. accurate estimates for plant's power demands,

determine a maintenance schedule respecting (2) that minimizes operating and maintenance costs.

## Example 2: Combinatorial Auctions

In **combinatorial auctions**, bidders can give bids for sets of items. The auctioneer then has to generate an optimal selection, e.g., one that maximizes revenue.

### Definition

The **combinatorial auction problem** is specified as follows:

**Given:** A set of items  $Q = \{q_1, \dots, q_n\}$  and a set of bids  $B = \{b_1, \dots, b_m\}$  such that each bid is  $b_i = (Q_i, r_i)$ , where  $Q_i \subseteq Q$  and  $r_i$  is a strictly positive real number.

**Task:** Find a subset of bids  $B' \subseteq B$  such that any two bids in  $B'$  do not share an item maximizing  $\sum_{(Q_i, r_i) \in B'} r_i$ .

# From Constraint to Cost Networks

- ▶ We will extend **constraint networks** to **cost networks**.
- ▶ **Hard constraint** are modelled as ordinary constraints, we know already.
- ▶ **Soft constraints** are modelled by **cost functions**, which assign particular costs to variable assignments.
- ▶ The costs are aggregated by a **global cost function**

## Global Cost Functions

A constraint optimization problem (COP) is a constraint network extended by a **global cost function**.

### Definition

Given a set of variables  $V = \{v_1, \dots, v_n\}$ , a set of real-valued functions  $F_1, \dots, F_l$  over scopes  $S_1, \dots, S_l$ ,  $S_j \subseteq V$ , and assignments  $a$  over  $V$ . The **global cost function**  $F$  is defined by

$$F(a) = \sum_{j=1}^l F_j(a),$$

where  $F_j(a)$  means  $F_j$  applied to assignments in  $a$  restricted to the scope of  $F_j$ , i.e.,  $F_j(a) = F_j(\bar{a}[S_j])$ .



# Cost Networks

Constraint optimization problems can be viewed as defined over an extended constraint network called **cost network**.

## Definition

A **cost network** is a 4-tuple  $\mathcal{O} = \langle V, \text{dom}, C_h, C_s \rangle$ , where  $\langle V, \text{dom}, C_h \rangle$  is a constraint network (elements of  $C_h$  are called **hard constraints**), and  $C_s = \{F_1, \dots, F_l\}$  is a set of real-valued functions defined over scopes  $S_1, \dots, S_l$  (elements of  $C_s$  are called **soft constraints**).

## Definition

A **solution to a constraint optimization problem** given by a cost network  $\mathcal{O} = \langle V, \text{dom}, C_h, C_s \rangle$ , is an assignment  $a^*$  that maximizes (minimizes)  $F(a)$  among all assignments  $a$  that satisfy  $\langle V, \text{dom}, C_h \rangle$ .

## Example: Cost Network for Combinatorial Auction

For a combinatorial auction given by item set  $Q = \{q_1, \dots, q_n\}$  and bids  $B = \{b_1, \dots, b_m\}$  with  $b_i = (Q_i, r_i)$  define a cost network as follows:

- ▶ **Variables**  $b_i$  with domain  $\{0, 1\}$ ; 1 for selecting the bid, 0 otherwise;
- ▶ For each pair  $b_i, b_j$  such that  $Q_i \cap Q_j \neq \emptyset$  a **constraint**  $R_{ij}$  prohibiting that  $b_i$  and  $b_j$  are assigned 1 simultaneously;
- ▶ **Cost functions**  $F_i$  with  $F_i(a) = r_i$  if  $a(b_i) = 1$ ,  $F_i(a) = 0$  otherwise, for an assignment  $a$ .

Find a consistent assignment  $a$  to the  $b_i$ s that **maximizes**

$$F(a) = \sum_i F_i(a).$$

Note: cost network = constraint network, because all cost components are unary.

## Example Auction

Consider the following auction:

$$\begin{aligned} b_1 &= \{1, 2, 3, 4\}, & r_1 &= 8, \\ b_2 &= \{2, 3, 6\}, & r_2 &= 6, \\ b_3 &= \{1, 4, 5\}, & r_3 &= 5, \\ b_4 &= \{2, 8\}, & r_4 &= 2, \\ b_5 &= \{5, 6\}, & r_5 &= 2. \end{aligned}$$

What is the optimal assignment?

## Reduction of COP-Solving to CSP-Solving

We can always reduce COP-solving to solving a **sequence of CSPs**.

Given a COP  $\mathcal{O}$  which we want to maximize. Consider a sequence of CSPs  $\mathcal{C}_i$ , s. t. each contains the constraint part of  $\mathcal{O}$  and an additional constraint  $\sum_j F_j(a) \geq c_i$ , where  $c_1 \leq \dots \leq c_i \leq \dots$

Solve the CSPs with increasing cost bounds  $c_i$  until no solution can be found. Then the previous step is the optimal solution – provided the difference between the steps is not larger than the smallest difference between different values of the global cost function.

As a variation, one could think of **binary search** approaches.

## Example: Solving the Auction Problem

Assumption: Step size 1 and static variable ordering  $b_1, b_2, b_3, b_4, b_5$ .

For cost bounds from  $c_1 = 0$  to  $c_9 = 8$ ,  $a(b_1) = 1$  and all others 0 is satisfying.

For cost bound  $c_{10} = 9$  and  $c_{11} = 10$ ,  $a(b_1) = 1$  and  $a(b_5) = 1$  (and all others 0) is satisfying.

For cost bound  $c_{12} = 11$ ,  $a(b_2) = 1$  and  $a(b_3) = 1$  (and all others 0) is satisfying.

For cost bound  $c_{13} = 12$ , there is no satisfying assignment.

## Branch and Bound: First idea

When solving a COP using a sequence of CSPs, one could use all CSP techniques. However, instead of solving multiple CSPs, one may instead want to integrate the optimization process into the search process.

First **idea**:

1. Set bound  $c = 0$ .
2. Use any systematic search technique to find an assignment that satisfies the constraint part.
3. Remember solution in  $a$  and global cost in  $c$  if global cost  $> c$ .
4. Return  $a$  and  $c$  if no further solutions can be found, otherwise continue with next solution at (3).

# Pruning

Of course, often it is possible to **prune** the search, even if no inconsistency has been detected yet.

Main idea behind **depth-first branch-and-bound (BnB)**:

If the best solution so far is  $c$ , this is a **lower bound** for all other possible solutions. So, if a partial solution has led to costs of  $x$  for all cost components of fully instantiated variables and **the best we can achieve** for all other cost components is  $y$  with  $x + y < c$ , then we do not need to continue in this branch.

How can we find out what is the best we can achieve?

## Bounding Evaluation Function

In the following, we will write  $\vec{a}_i$  for partial instantiations of the first  $i$  variables, assuming a static variable ordering.

### Definition

A **bounding evaluation function** for a **maximizing (minimizing) constraint optimization problem** is a function  $f$  over partial assignments such that  $f(\vec{a}_i) \geq \max_a F(a)$  ( $f(\vec{a}_i) \leq \min_a F(a)$ ) for all satisfying assignments  $a$  that extend  $\vec{a}_i$ .

Note:

- ▶ If  $f(\vec{a}_i) < c$  for some already found solution  $c$ , then  $\vec{a}_i$  cannot be extended to a maximal solution.
- ▶  $f$  can also be used as a heuristic for choosing a value of the next variable!



# Branch and Bound (BnB) Algorithm

**BnB**( $\mathcal{O}, f$ ):

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*Input:* cost network  $\mathcal{O}$  and evaluation bounding function  $f$

*Output:* an optimal assignment  $a'$  (possibly empty) and costs  $c'$

$\forall i D'_i \leftarrow D_i, i \leftarrow 1, c' \leftarrow 0, a' \leftarrow \emptyset, a \leftarrow \emptyset$

**while** ( $i \neq 0$ )

**while** ( $1 \leq i \leq n$ )

        remove ( $v_i \mapsto \_$ ) from  $a$  // remove old assignment to  $v_i$

$x \leftarrow \text{SELECTVALUE}(i, c')$

**if** ( $x = \text{null}$ )  $D'_i \leftarrow D_i$  // no value for  $x_i$ : reset domain

$i \leftarrow i - 1$  // backtrack

**else**  $a \leftarrow a \cup \{v_i \mapsto x\}$

$i \leftarrow i + 1$  // step forward

**if** ( $i = n + 1$ ) // one solution found

**if** ( $F(a) > c'$ ) // better solution

$a' \leftarrow a$  // remember best solution found so far

$c' \leftarrow F(a)$

$i \leftarrow n$  // search for next solution

**return**( $a', c'$ )

# Branch and Bound Algorithm: SELECTVALUE

SELECTVALUE( $i, c'$ ):

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**while** ( $D'_i \neq \emptyset$ )  
     select  $a_i^* \in D'_i$  such that  
          $a_i^* = \text{pick one arg max}_{a_i \in D'_i} f(a \cup \{v_i \mapsto a_i\})$   
     remove  $a_i^*$  from  $D'_i$   
     **if** ( $a \cup \{v_i \mapsto a_i^*\}$ ) is consistent **and**  
          $f(a \cup \{v_i \mapsto a_i^*\}) > c'$  **return**( $a_i^*$ )  
**return**(*null*)

## First-Choice Bounding Function

How to come up with a good **bounding evaluation function**?

In *Operation Research*, one often uses **Linear Programming** to come up with bounds for **Integer Programming Problems**.

Let us consider what we can achieve for all soft constraints in isolation subject to the partial assignment we have already. This function is called **first-choice** (*fc*) bounding function:

$$f_{fc}(\vec{a}_i) = \sum_{F_j \in C_s} \max_{a_{i+1}, \dots, a_n} F_j(\vec{a}_i \cup \{v_{i+1} \mapsto a_{i+1}, \dots, v_n \mapsto a_n\})$$

How could one improve on that?

- ▶ Only allow locally consistent partial assignments.
- ▶ Do not consider all soft constraints in isolation, but combine them!

## Example: Auction again

Let us consider **BnB** with the **first-choice** bounding function on our auction example:

1.  $f_{fc}(\{b_1 \mapsto 1\}) = 8 + (6 + 5 + 2 + 2) = 23$
2.  $f_{fc}(\{b_1 \mapsto 1, b_2 \mapsto 0\}) = 8 + (5 + 2 + 2) = 17$
3.  $f_{fc}(\{b_1 \mapsto 1, b_2 \mapsto 0, b_3 \mapsto 0\}) = 8 + (2 + 2) = 12$
4.  $f_{fc}(\{b_1 \mapsto 1, b_2 \mapsto 0, b_3 \mapsto 0, b_4 \mapsto 0\}) = 8 + (2) = 10$
5. ...

## Russian Doll Search: Idea

One way to get more accurate bounding functions is to solve subproblems and store the optimal results, reusing them for larger problems.

Solve a sequence of  $n$  problems using BnB, where in the  $i$ th run the last  $i$  variables, i.e.,  $v_{n-i+1}$  up to  $v_n$ , (and the relevant hard and soft constraints) are considered.

The results of the previous runs can be used:

1. as an initial lower bound,
2. in a heuristic for choosing values, and
3. to generate a more accurate bounding function.

## Improving the Evaluation Function

- ▶ Solve  $n$  COPs  $\mathcal{O}_i$ , ( $i = 1, \dots, n$ ) over the last  $i$  variables  $v_{n-i+1}, \dots, v_n$  using BnB and store maximal costs as  $c_i^*$ .
- ▶ In the  $(n - i + 1)$ th run, variables  $v_i, \dots, v_n$  are considered.
- ▶ Assume that the variables  $v_i, \dots, v_{i+j}$  are instantiated, denoted by the partial assignment  $\vec{a}_j^i$ , and that  $C_{i,j}$  are all those soft constraints  $F$  such that their scopes have a non-empty intersection with  $\{v_i, \dots, v_{i+j}\}$ .
- ▶ Then we use the optimal costs from the  $n - i - j$ th run to improve on the first-choice function:

$$f(\vec{a}_j^i) = c_{n-i-j}^* + \sum_{F \in C_{i,j}} \max_{a_{i+j+1}, \dots, a_n} F(\{v_i \mapsto a_i, \dots, v_n \mapsto a_n\}).$$

## Conclusion & Outlook

- ▶ Problems with hard and soft constraints lead to **constraint optimization problems**
- ▶ These are formalized using **cost functions** and **cost networks**
- ▶ They can be solved using a reduction to a sequence of CSP problems
- ▶ More efficiently, one can search for optimal solutions during the backtracking search
- ▶ **Branch and Bound** is the method of choice
- ▶ Its pruning power depends on the accuracy of the **bounding evaluation function**
- ▶ **Russian doll search** can boost its performance
- ▶ Further enhancements are possible using **constraint inference techniques** (such as **bucket elimination**).

# Literature



Rina Dechter.  
Constraint Processing,  
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