Constraint Satisfaction Problems

Global Constraints

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Global Constraints

What are global Constraints?

- Type of similar constraint relations . . .
- . . . differing in the number of variables
- Semantically redundant: same constraint can be expressed by a conjunction of simpler constraints
- Similar structure: can be exploited by constraint solvers

Examples:

- sum constraint, knapsack constraint, element constraint, all-different constraint, cardinality constraints

All-different constraint

Definition

Let $v_1, \ldots, v_n$ be variables each with a domain D_i ($1 \leq i \leq n$).

$$\texttt{alldifferent}(v_1, \ldots, v_n) := \\ \{(d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n \;:\; d_i \neq d_j \text{ for } i \neq j\}$$

The all-different constraint is a simple, but widely used global constraint in constraint programming.
It allows for compact modeling of CSP problems.

Example: n-Queens Problem

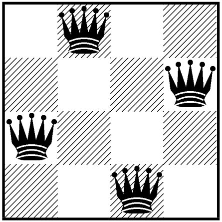


Figure: 4-queens problem

Problem representation: Variables v_i for each column $1, \ldots, n$; v_i can take a "row value" $1, \ldots, n$.

No-attack constraints:

$$v_i \neq v_j \text{ for } 1 \leq i < j \leq n$$
$$v_i - v_j \neq i - j \text{ for } 1 \leq i < j \leq n$$
$$v_j - v_i \neq j - i \text{ for } 1 \leq i < j \leq n$$

$$\texttt{alldifferent}(v_1, \ldots, v_n)$$
$$\texttt{alldifferent}(v_1 - 1, \ldots, v_n - n)$$
$$\texttt{alldifferent}(v_1 + 1, \ldots, v_n + n)$$

Sum Constraint

Let $v_1, \ldots, v_n, z$ be variables with subsets of $\mathbb{Q}$ as domain.
For each v_i, let $c_i \in \mathbb{Q}$ be some fixed scalar, $c = (c_1, \ldots, c_n)$.

Definition

The sum constraint is defined as:

$$\mathtt{sum}(v_1, \ldots, v_n, z, c) := \Big\{(d_1, \ldots, d_n, d) \in \big(\prod_{1 \leq i \leq n} D_i\big) \times D_z \;:\; d = \sum_{1 \leq i \leq n} c_i d_i \Big\}.$$

Global Cardinality Constraint

$v_1, \ldots, v_n$: "assignment variables" with $D_i \subseteq \{d_1^*, \ldots, d_m^*\}$.
$c_1, \ldots, c_m$: "count variables" with sets of integers as domains.

Definition

The global cardinality constraint is defined as:

$$\mathrm{gcc}(v_1, \ldots, v_n, c_1, \ldots, c_m) := \Big\{(d_1, \ldots, d_n, o_1, \ldots, o_m) \in \prod_{1 \leq i \leq n} D_{v_i} \times \prod_{1 \leq j \leq m} D_{c_j} \; : \text{for each } j,\ d_j^* \text{ occurs in } (d_1, \ldots, d_n) \text{ exactly } o_j \text{ times}\Big\}$$

The global cardinality constraint can be considered a generalization of the all-different constraint.

Circuit Constraint

Let $s = (s_1, \ldots, s_n)$ be a permutation of $\{1, \ldots, n\}$.
Define C_s as the smallest set that contains 1 and with each element i also s_i.
$(s_1, \ldots, s_n)$ is called cyclic if $C_s = \{1, \ldots, n\}$.

Definition

Let $v_1, \ldots, v_n$ be variables with domains $D_i = \{1, \ldots, n\}$ $(1 \leq i \leq n)$.

$$\texttt{circuit}(v_1, \ldots, v_n) := \Big\{(d_1, \ldots, d_n) \in D_1 \times \cdots \times D_n \;:\; (d_1, \ldots, d_n) \text{ is cyclic}\Big\}$$

Given an assignment $a = (d_1, \ldots, d_n)$, define

$$A := \{(v_i, v_{d_i}) : d_i \in D_i, 1 \leq i \leq n\}\,.$$

Then, a satisfies $\texttt{circuit}(v_1, \ldots, v_n)$ if and only if (V, A) is a directed cycle (without proper sub-cycles).

Example: Traveling Salesperson Problem

Traveling Salesperson Problem (TSP):

Given a set of n cities and distances c_{ij} between city i and city j, find the shortest route that visits all cities and finishes in the starting city.

TSP is not a constraint satisfaction problem, but a constraint optimization problem . . .

Constraint Optimization Problem

Definition

A constraint optimization problem (COP) is a constraint satisfaction problem together with an objective function f that assign to each variable assignment a a value $f(a) \in \mathbb{Q}$.

- Minimization COP: Find a solution a that minimizes $f(a)$.
- Maximization COP: Find a solution a that maximizes $f(a)$.
- Optimal solution: Solution to a minimization (maximization) COP.

Decision problem associated to a COP:

Given an instance of a COP, (P, f), and some threshold $t \in \mathbb{Q}$, is there a solution a of P such that $f(a) \geq t$ ($f(a) \leq t$, resp.)?

The Decision Problem of TSP

v_i : variable for city i with domain $D_i := \{1, \ldots, n\} \setminus \{i\}$
(read as: value of v_i is the city to be visited next)

c_{ij} : distance between cities i and j (may not be symmetric)

t : bound for the total tour length

Then:

$$\texttt{circuit}(v_1, \ldots, v_n)$$

$$\sum_{1 \leq i \leq n} c_{iv_i} \leq t$$

Filtering

- Constraint propagation techniques aim at filtering variable domains: remove useless values (that cannot participate in any solution) as early as possible.
- Filtering allows false-positives (values are kept though they are useless),
- . . . ... but not false-negatives (useful value is removed).
- A constraint is “good” if it allows significant filtering (pruning of domain values) with low computational efforts.
- Constraint solver may benefit from exploiting the structure of such good constraints.

Filtering

Let (s, R) be a constraint.

Filtering algorithm: a filtering algorithm for a constraint (s, R) is an algorithm that filters the domains with respect to (s, R)

Complete filtering: every useless value from the domain of every variable that C is defined on is removed

Partial filtering: incomplete filtering

Enforcing Arc Consistency as Filtering Method

- In general, enforcing generalized arc consistency on a constraint network requires exponential time w.r.t. the largest arity of some constraint relation in the network.

  Recall: Enforcing generalized arc consistency runs in time

  $$O(erd^r),$$

  where e is the number of constraints and r is the largest arity of some constraint in the network,
- Though general constraints have often high arity, there exist efficient methods to enforce generalized arc consistency.
- In the following we consider the all-different constraints.

Value Graphs

Definition

An undirected graph $G = \langle V, E \rangle$ is bipartite if there exists a partition $S \dot{\cup} T$ of V such that $E \subseteq S \times T$.

A directed graph $G = \langle V, A \rangle$ is bipartite if there exists a partition $S \dot{\cup} T$ of V such that $A \subseteq (S \times T) \cup (T \times S)$.

G is then written in the form $G = \langle S, T, E \rangle$ / $G = \langle S, T, A \rangle$.

Definition

Let V be a set of variables and D be the union of all domains D_v for $v \in V$.

The value graph of V is defined as the following bipartite graph:

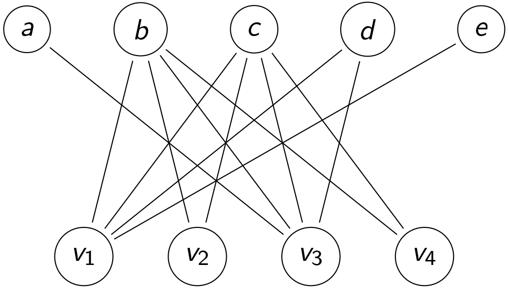
$$G = \langle V, D, E \rangle$$

where $E = \{\{v, d\} : v \in V, d \in D_v\}$.

Example: Value graph

Consider variables $v_1, \ldots, v_4$ with $D_1 = \{b, c, d, e\}$, $D_2 = \{b, c\}$, $D_3 = \{a, b, c, d\}$, $D_4 = \{b, c\}$.

Value graph:



Matchings

Let $G = \langle V, E \rangle$ be an undirected graph.

Definition

A matching in G is a set $M \subseteq E$ of pairwise disjoint edges.
A matching M covers a set $S \subseteq V$ if $S \subseteq \bigcup M$, i.e., each $v \in S$ is contained in some edge in M.
$v \in V$ is M-free if M does not cover $\{v\}$.

Cardinality of a matching M: number of edges in M.

Definition

A path $v_0, \ldots, v_k$ in G is M-alternating if all the edges $\{v_i, v_{i+1}\}$ are alternatingly out of and in M.
A path $v_0, \ldots, v_k$ is M-augmenting if k is odd, M does not cover v_0 and v_k, and its edges $\{v_i, v_{i+1}\}$ are alternatingly out of and in M .

Let $G = \langle V, E \rangle$ be a graph and M be a matching in G.

Theorem (Peterson)

M is a max-cardinality matching (i.e., it is a matching of maximum cardinality) if and only if there is no M-augmenting path in G.

Hence a max-cardinality matching can be obtaind if one repeatedly searches for an M-augmenting path in G and uses it to extend M.

Note: If M is a matching and $v_0, \ldots, v_k$ is an M-augmenting path, then

$$M' := M \oplus \{\{v_i, v_{i+1}\} : 0 \leq i \leq k-1\}$$

is a matching with $|M'| = |M| + 1$.

Max-Cardinality Matching on Bipartite Graphs

Let $G = \langle U, W, E\rangle$ be a bipartite graph and M be some matching. We may assume $|U| \leq |W|$.
Define a directed bipartite graph $G_M = \langle U, W, A\rangle$ by

$$A := \big\{(w,u) \;:\; \{u,w\} \in M, u \in U, w \in W\big\} \cup \\ \big\{(u,w) \;:\; \{u,w\} \in E \setminus M, u \in U, w \in W\big\}$$

Every directed path in G_M starting in an M-free vertex in U and ending in an M-free vertex in W corresponds to an M-augmenting path in G.
We need to find at most $|U|$ such paths.
Each path can be identified by breadth-first search in time $O(|A|)$.

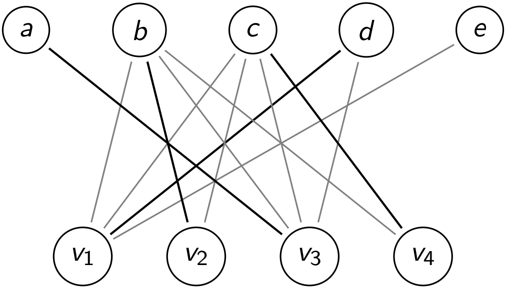
This method by van der Waerden and König can be improved by an algorithm by Hopcroft and Karp ($O(\sqrt{|U|} \cdot |A|)$).

All-different Constraint and Matching

Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and G be the value graph of V.
Let $(d_1, \ldots, d_n)$ be a variable assignment.

Lemma

$(d_1, \ldots, d_n) \in \mathit{alldifferent}(v_1, \ldots, v_n)$ *if and only if* $M = \{\{v_1, d_1\}, \ldots, \{v_n, d_n\}\}$ *is a matching in* G.



Arc-consistent All-different Constraint

Lemma

The constraint $\mathtt{alldifferent}(v_1, \ldots, v_n)$ is generalized arc-consistent, if and only if every edge in G belongs to a matching in G that covers V.

Proof.

Simple. □

Edges in Max-Cardinality Matchings

Theorem

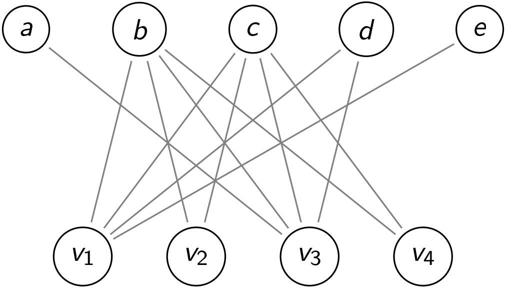
Let G be a graph and let M be a max-cardinality matching in G. An edge e belongs to some max-cardinality matching in G if and only if one of the following conditions holds:

- $e \in M$.
- *e is on an even-length M-alternating path starting at an M-free vertex;*
- *e is on an even-length M-alternating circuit.*

Enforcing Arc Consistency on All-different Constraints

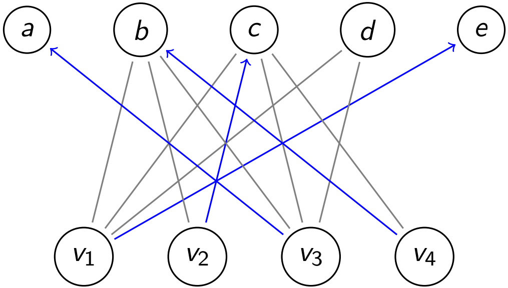
1. Compute a max-cardinality matching M in the value graph of V (can be done in time $O(m\sqrt{n})$ where $m = \sum_{1 \leq i \leq n} |D_i|$)
2. Identify the even M-alternating paths starting in an M-free vertex and the M-alternating cycles:
   2.1 Define dir. bipartite graph $G_M = \langle V, D_V, A \rangle$ with $A = \{(v, d) : v \in V, \{v, d\} \in M\} \cup \{(d, v) : v \in V, \{v, d\} \in E \setminus M\}$
   2.2 Compute the strongly connected components in G_M (in time $O(n + m)$)
   2.3 Mark acrs between vertices in the same component as "used": they belong to an even M-alternating cycle
   2.4 Marc arcs as "used" that belong to a directed path in G_M, start in an M-free vertex (breadth-first search in time $O(m)$).
3. Update $D_v \leftarrow D_v \setminus \{d\}$ for all edges $\{v, d\}$ where the corresponding arc is not marked as used.

Example: Enforcing Arc-Consistency



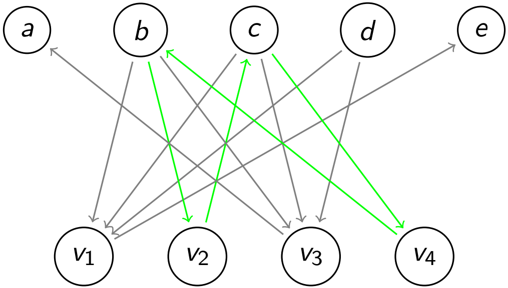
1. Compute max-cardinality matching
   $M = \{\{v_4, b\}, \{v_2, c\}, \{v_1, e\}, \{v_3, a\}\}$

Example: Enforcing Arc-Consistency



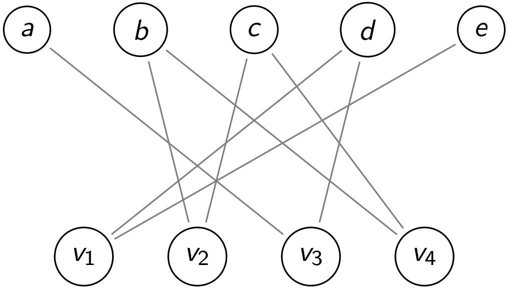
2. Identify supported values:
   (a) Identify G_M (b) Compute strongly connected components
      (e.g. by Kosaraju's algorithm)
   (c) Mark "used" arcs (d is the only M-free vertex)

Example: Enforcing Arc-Consistency



3. Filter unsupported values:
   Remove unused arcs

Example: Enforcing Arc-Consistency



Literature

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