

# Constraint Satisfaction Problems

## Tractable Constraint Languages

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based on a slideset by  
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# Expressiveness vs. Complexity

- For some restricted constraint languages we know some polynomial time algorithms that solve each instance of that language
  - Restricting constraint languages entails restricting expressiveness, i.e., the class of problems that can be expressed in the language
- ~> How can we weight expressiveness against performance and vice versa?

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# CSP Instances aka Constraint Networks

## Definition

An **instance of a constraint satisfaction problem** (i.e., a **constraint network**) is a triple

$$P = \langle V, D, C \rangle,$$

where:

- $V$  is a non-empty and finite set of **variables**,
- $D$  is an arbitrary set (**domain**),
- $C$  is a finite set of **constraints**  $C_1, \dots, C_q$ , i.e., each constraint  $C_i$  is a pair  $(s_i, R_i)$ , where  $s_i$  is a tuple of variables of length  $m_i$  and  $R_i$  is an  $m_i$ -ary relation on  $D$  ( $s_i$ : **constraint scope**;  $R_i$ : **constraint relation**).

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# Restricting the General CSP

The **general CSP decision problem** is the following: Given an instance of a constraint satisfaction problem,  $P$ , determine if there exists solution to  $P$ , i.e., determine whether

$$\text{Sol}(P) \\ := \{(d_1, \dots, d_n) \in D^n : a(v_i) = d_i \text{ for a solution } a \text{ of } P\}$$

(where  $n$  is the number of variables of  $V$ ) is not empty.

Restricting the general CSP:

- **structural restriction**: consider just CSP instances with particular constraint scopes (e. g., where the network hypergraph has specific properties)
- **relational restriction**: consider just CSP instances, where the constraint relations have a specific form or specific properties

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# Constraint Language

## Definition

A **constraint language** is an arbitrary set of relations,  $\Gamma$ , defined over some fixed domain (denoted by  $D(\Gamma)$ ).

## Definition

For a constraint language  $\Gamma$ , let  $\text{CSP}(\Gamma)$  be the class of CSP instances  $P = \langle V, D, C \rangle$  such that for each  $(s, R) \in C$ ,  $R \in \Gamma$ .  $\text{CSP}(\Gamma)$  is called the **relational subclass** associated with  $\Gamma$ .

## Definition

A finite constraint language  $\Gamma$  is **tractable** if there exists a polynomial algorithm that solves all instances of  $\text{CSP}(\Gamma)$ .  
An infinite constraint language  $\Gamma$  is **tractable** if each finite subset of the language is tractable.

Following, we present some examples:

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# Example: the CHIP language

CHIP is a constraint language for arithmetic and other constraints.

Basic constraints in CHIP are so-called:

- **domain constraints**: unary constraints that restrict the domains of variables to a finite set of natural numbers
- **arithmetic constraints**: constraints of one of the forms

$$ax = by + c$$

$$ax \leq by + c$$

$$ax \geq by + c$$

( $a, b, c \in \mathbb{N}, a \neq 0$ ). If these equations are conceived of as relations, the resulting constraint language is tractable.

The language is still tractable if we allow for relations expressed by

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \geq by + c$$

$$ax_1 \dots x_n \geq by + c$$

$$(a_1x_1 \geq b_1) \vee \dots \vee (a_nx_n \geq b_n) \vee (ay \geq b)$$

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# Example: Linear Relations

Let  $D$  be any field (e.g., the field of real numbers).

A **linear relation** on  $D$  is any relation defined by some system of linear equations:

$$a_1x_1 + \cdots + a_nx_n = r \quad (a_1, \dots, a_n, r \in D).$$

Then any instance of  $\text{CSP}(\Gamma_{\text{lin}})$  can be represented by a system of linear equations over  $D$ , and hence be solved in polynomial time (apply Gaussian elimination).

Hence, the language of all linear relations over  $D$  is tractable.

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# Example: Relations on Ordered Finite Sets

Let  $D$  be an ordered and finite set.

Consider the binary **disequality relation**

$$\neq_D = \{(d_1, d_2) \in D^2 : d_1 \neq d_2\}$$

The class of CSP instances  $\text{CSP}(\{\neq_D\})$  corresponds to the graph colorability problem with  $|D|$  colors.

$\text{CSP}(\{\neq_D\})$  is tractable if  $|D| \leq 2$ , and intractable, otherwise.

The ternary **betweenness relation** over  $D$  is defined by:

$$B_D = \{(a, b, c) \in D^3 : a < b < c \vee c < b < a\}$$

$\text{CSP}(\{B_D\})$  is tractable if  $|D| \leq 4$ , and intractable if  $|D| \geq 5$ .

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# Example: Connected Row-Convex Relations

Let  $D = \{d_1, \dots, d_n\}$  be an ordered and finite set.

For a binary relation  $R$  over  $D$ , the matrix representation of  $R$  is an  $n \times n$  0,1-matrix  $M$ , where  $M_{ij} = 1$  iff  $(d_i, d_j) \in R$ .

The **pruned matrix representation** of  $R$  results from the matrix representation of  $R$ , when we remove all rows and columns in which only 0's occur.

$R$  is **connected row-convex**, if in the pruned matrix representation of  $R$ , the pattern of 1's is connected along each column, along each row, and forms a connected 2-dimensional region.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The constraint language on any class of connected row-convex relations is tractable.

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# Example: Boolean Constraints

Let  $D = \{d_0, d_1\}$ .

The class of CSP instances  $\text{CSP}(\{N_D\})$ , where

$$N_D = D^3 \setminus \{(d_0, d_0, d_0), (d_1, d_1, d_1)\}$$

is the **not-all-equal relation** over  $D$ , is intractable.

$\text{CSP}(\{N_D\})$  corresponds to the not-all-equal satisfiability problem (NAE-3SAT), which is known to be NP-hard.

The class of CSP instances  $\text{CSP}(\{T_D\})$ , where

$$T_D = \{(d_0, d_0, d_1), (d_0, d_1, d_0), (d_1, d_0, d_0)\},$$

is intractable.

$\text{CSP}(\{T_D\})$  corresponds to the one-in-three satisfiability problem (1-in-3 SAT).

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## Example: 0/1/all-Relations

Let  $D$  be an arbitrary finite set. A relation  $R$  over  $D$  is called **0/1/all-relation** if one of the following conditions holds:

- $R$  is unary;
- $R = D_1 \times D_2$  for subsets  $D_1, D_2$  of  $D$ ;
- $R = \{(d, \pi(d)) : d \in D_1\}$ , for some subset  $D_1 \subseteq D$  and some permutation  $\pi$  of  $D$ ;
- $R = \{(a, b) \in D_1 \times D_2 : a = d_1 \vee b = d_2\}$ , for some subsets  $D_1, D_2$  of  $D$  and some elements  $d_1 \in D_1, d_2 \in D_2$ .

The language defined by all 0/1/all-relations is tractable.

It is even **maximal tractable**: if we add any binary relation over  $D$  that is not a 0/1/all-relation, then the resulting constraint language becomes intractable.

# max-Closed Relations

Let  $(D, <)$  be a linear order. Define  $\max : D \times D \rightarrow D$  in the usual way, i.e.,  $\max(a, b) = a$  if  $a > b$ , and  $\max(a, b) = b$ , otherwise.

We extend  $\max$  to a function that can be applied to tuples, i.e., we define  $\max : D^k \times D^k \rightarrow D^k$  by

$$\begin{aligned} \max((a_1, \dots, a_k), (b_1, \dots, b_k)) \\ := (\max(a_1, b_1), \dots, \max(a_k, b_k)). \end{aligned}$$

## Definition

An  $n$ -ary relation  $R$  over  $D$  is **max-closed** if for all  $(a_1, \dots, a_n), (b_1, \dots, b_n) \in R$ ,

$$\max((a_1, \dots, a_n), (b_1, \dots, b_n)) \in R.$$

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# max-Closed Relations and Tractability

## Lemma

*Let  $\Gamma$  be a constraint language with max-closed relations only. Then  $CSP(\Gamma)$  is tractable.*

## Proof.

Enforce generalized arc consistency. If any domain of the resulting network is empty, the network is inconsistent. Otherwise, set each variable to its maximal value, . . . . □

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## Example: max-Closed Relations

Consider the CHIP language. All relations of CHIP are max-closed. Hence any set of equations can be solved by establishing arc consistency.

For example, consider a CSP instance with domain  $\{1, \dots, 5\}$ , variables  $\{v, w, x, y, z\}$ , and equations

$$w \neq 3, \quad z \neq 5, \quad 3v \leq z, \quad y \geq z + 2, \\ 3x + y + z \geq 5w + 1, \quad wz \geq 2y.$$

Enforcing arc consistency results in:

$$D(v) = \{1\}, \quad D(w) = \{4\}, \quad D(x) = \{3, 4, 5\}, \\ D(y) = \{5\}, \quad D(z) = \{3\}.$$

Hence

$$v \mapsto 1, w \mapsto 4, x \mapsto 5, y \mapsto 5, z \mapsto 3$$

is a solution of the constraint network.

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# Boolean Constraint Languages

The key result in the literature on tractable constraint languages is Schaefer's Dichotomy Theorem (1978).

## Definition

A **Boolean constraint language** is a constraint language over the two-element domain  $D = \{0, 1\}$ .

Schaefer's theorem states that any Boolean constraint language is either tractable or NP-complete. Moreover, it provides a classification of all tractable constraint languages.

## Definition

An arbitrary constraint language  $\Gamma$  is **NP-complete** if  $\text{CSP}(\Delta)$  is NP-complete for some finite subset  $\Delta \subseteq \Gamma$ .

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# Schaefer's Theorem

## Theorem (Schaefer 1978)

*Let  $\Gamma$  be a Boolean constraint language. Then  $\Gamma$  is tractable if at least one of the following conditions is satisfied:*

- 1 *Each relation in  $\Gamma$  contains the tuple  $(0, \dots, 0)$ .*
- 2 *Each relation in  $\Gamma$  contains the tuple  $(1, \dots, 1)$ .*
- 3 *Each relation in  $\Gamma$  is definable by a formula in CNF s. t. each conjunct has at most one negative literal.*
- 4 *Each relation in  $\Gamma$  is definable by a formula in CNF s. t. each conjunct has at most one positive literal.*
- 5 *Each relation in  $\Gamma$  is definable by a formula in CNF s. t. each conjunct has at most two literals.*
- 6 *Each relation in  $\Gamma$  is the set of solutions of a system of linear equations over the finite field with 2 elements.*

*In all other cases,  $\Gamma$  is NP-complete.*

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# Algorithm Selector

Let  $\Gamma$  be a Boolean constraint language.

- Class 1:** any CSP instance  $P$  can be solved by simply assigning 0 to each variable of  $P$ .
- Class 2:** cf. Class 1 ( $v \mapsto 1$ ).
- Class 6:** any CSP instance  $P$  can be solved by applying the Gaussian elimination procedure.
- Class 5:** any CSP instance  $P$  can be solved by resolution: in this case  $\text{CSP}(\Gamma)$  corresponds to the 2-SAT satisfiability problem and this can be solved efficiently by resolution.
- Class 4:** any CSP instance  $P$  can be solved by unit resolution: here  $\text{CSP}(\Gamma)$  corresponds to the Horn-SAT satisfiability problem, which can be solved efficiently by unit resolution.
- Class 3:** cf. Class 4 (“anti-Horn”).

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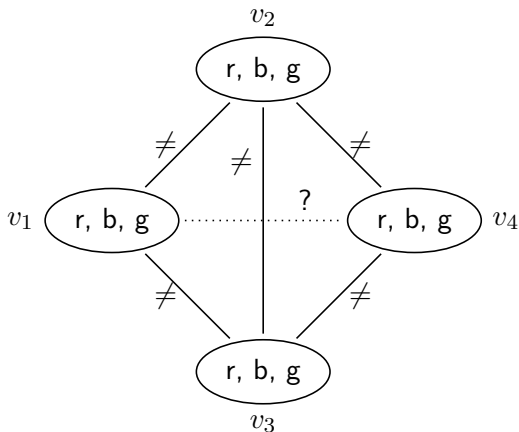
## Definition

Let  $\Gamma$  be constraint language and  $R$  be a relation on  $\Gamma(D)$ .  
 $R$  is **expressible** in  $\Gamma$  if there exists a CSP instance  $P \in \text{CSP}(\Gamma)$   
and a sequence of variables  $v_1, \dots, v_n$  such that

$$R = \pi_{v_1, \dots, v_n}(\text{Sol}(P)).$$

$P$  is referred to as a **gadget** for expressing  $R$  in  $\text{CSP}(\Gamma)$ , the  
sequence  $v_1, \dots, v_n$  as **construction site** for  $R$ .

# Example



Which relation is expressed by the edge  $(v_1, v_4)$ ?

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# Relational Clones

Expressiveness can also be reformulated in the following way:  
Let  $\Gamma, \Gamma'$  be constraint languages (def. on the same domain  $D$ ).

## Definition

$\Gamma'$  is a **relational clone** of  $\Gamma$  if  $\Gamma'$  contains each relation expressible by a FO-formula with

- relations from  $\Gamma \cup \{=_D\}$ ,
- conjunctions, and
- existential quantification.

(Formulae of this form are called **primitive positive formulae**.)

## Definition

Let  $\Gamma$  be a constraint language.  $\langle \Gamma \rangle$  denotes the smallest relational clone containing  $\Gamma$ , **the clone generated by  $\Gamma$** .

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# Example

Consider a Boolean constraint language with the following relations:

$$R_1 = \{(0, 1), (1, 0), (1, 1)\} \quad R_2 = \{(0, 0), (0, 1), (1, 0)\}.$$

The relational clone generated by the set of these two relations contains all 16 binary Boolean relations. For example:

$$\begin{array}{ll} R_3 := \{(0, 1), (1, 0)\} & R_1(v_1, v_2) \wedge R_2(v_1, v_2) \\ R_4 := \{(0, 0), (1, 0), (1, 1)\} & \exists y (R_1(v_1, y) \wedge R_2(y, v_2)) \\ R_5 := \{(0, 0), (1, 1)\} & v_1 = v_2 \\ R_6 := \{(0, 0)\} & R_2(v_1, v_2) \wedge R_5(v_1, v_2) \\ R_7 := \{(1, 1)\} & R_1(v_1, v_2) \wedge R_5(v_1, v_2) \\ R_8 := \{(0, 1)\} & \exists y (R_6(v_1, y) \wedge R_1(y, v_2)) \end{array}$$

...

## Theorem

*Let  $\Gamma$  be a set of relations on a fixed domain  $D$ , and let  $\Delta$  be a finite subset of  $\langle \Gamma \rangle$ . Then there exists a polynomial time reduction from  $CSP(\Delta)$  to  $CSP(\Gamma)$ .*

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# Reducibility II

## Proof.

Let  $\Delta = \{S_1, \dots, S_k\}$  be a finite set of relations, where each  $S_j$  is expressible by a pp-formula with relations from  $\Gamma$  and the relation  $=_D$ . For each  $S_j$  fix such a formula  $\phi_j(x_1, \dots, x_{r_j})$ , where  $r_j$  is the arity of  $S_j$ . Without loss of generality, we may assume that each  $\phi_j(x_1, \dots, x_{r_j})$  has the form

$$\exists u_1 \dots u_m (R_1(w_1^1, \dots, w_{k_1}^1) \wedge \dots \wedge R_n(w_1^n, \dots, w_{k_n}^n)) \quad (1)$$

where  $w_1^1, \dots, w_{k_1}^1, \dots, w_1^n, \dots, w_{k_n}^n \in \{x_1, \dots, x_{r_j}, u_1, \dots, u_m\}$  for some auxiliary variables  $u_1, \dots, u_m$ , and  $R_1, \dots, R_n \in \Gamma \cup \{=_D\}$ .

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# Reducibility III

Let  $P = \langle V, D, C \rangle$  be an arbitrary instance in  $\text{CSP}(\Delta)$ . Initially, set  $V' := V, D' := D, C' := C$ . For each constraint  $(s, R)$  (where  $s = (v_1, \dots, v_r)$ ) of  $P$ , proceed as follows:

- 1 add the auxiliary variables  $u_1, \dots, u_m$  to  $V'$  (always add new variables, rename variables if necessary (also in (1)))
- 2 remove  $(r, R)$  from  $C'$  and instead add to  $C'$  the constraints (cf. (1)):

$$((w_1^1, \dots, w_{k_1}^1), R_1), \dots, (w_1^n, \dots, w_{k_n}^n, R_n)$$

The CSP instance  $P'$  obtained by this procedure is contained in  $\text{CSP}(\Gamma \cup \{=_D\})$  and is obviously equivalent to  $P$ . Furthermore, from  $P'$  we can obtain a CSP instance  $P''$  in  $\text{CSP}(\Gamma)$  by deleting constraints of the form  $((v_i, v_j), =_D)$  and replacing any occurrence of  $v_j$  by  $v_i$ . Obviously, both transformation can be done in polynomial time.  $\square$

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# The Indicator Problem

Let  $k \geq 1$  be a fixed natural number.

Let  $s = (x_1, \dots, x_m)$  be a list of  $k$ -tuples in  $D^k$ .

Let  $R$  be an  $n$ -ary relation on  $D$ .

We say, that  $s$  **matches**  $R$  if  $n = m$  and if for each  $1 \leq i \leq k$ , the  $n$ -tuple  $(x_1[i], \dots, x_n[i])$  is in  $R$ .

Let now  $\Gamma$  be a fixed constraint language. Set  $I_k(\Gamma) = \langle V, D, C \rangle$ , where

$$V := D^k$$

$$C := \{(s, R) : s \text{ matches } R\}$$

Note:  $I_k(\Gamma) \in \text{CSP}(\Gamma)$  and contains constraints from  $\Gamma$  on every possible scope which matches some relation in  $\Gamma$ .

## Definition

$I_k(\Gamma)$  is said to be the **indicator problem of order  $k$**  for  $\Gamma$ .

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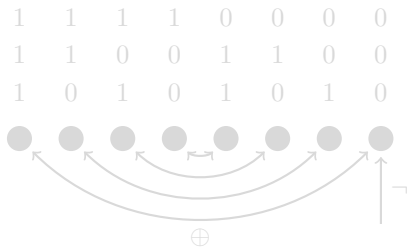
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# Example: $\neg, \oplus$

Consider the Boolean constraint language containing the unary relation  $\neg$  and the exclusive-or relation  $\oplus$ , i.e.,

$$R_{\oplus} = \{(0, 1), (1, 0)\} \quad \text{and} \quad R_{\neg} = \{(0)\}.$$

The 3-rd order indicator problem of this language is:

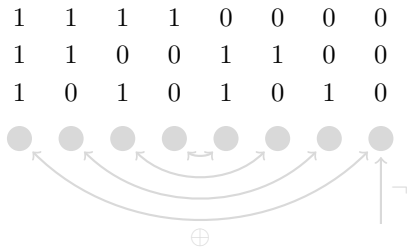


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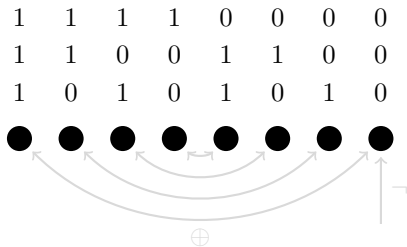


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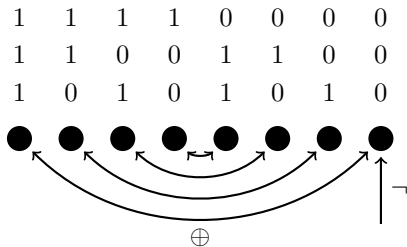


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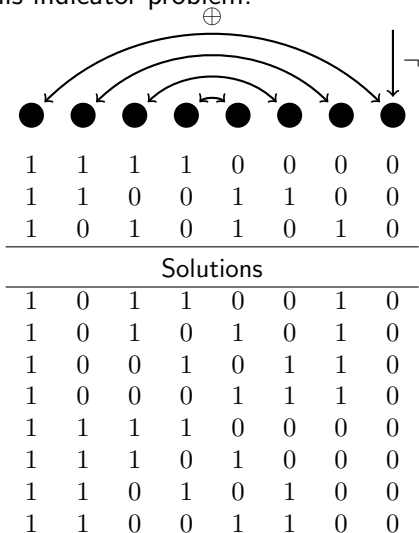
$$R_{\oplus} = \{(0, 1), (1, 0)\} \quad \text{and} \quad R_{\neg} = \{(0)\}.$$

The 3-rd order indicator problem of this language is:



# Example (cont'd): $\neg, \oplus$

Solutions of this indicator problem:



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# Expressiveness and the Indicator Problem

## Theorem (Jeavons (1998))

Let  $\Gamma$  be a constraint language over some finite domain  $D$  and let  $R = \{t_1, \dots, t_k\}$  be any  $n$ -ary relation on  $D$ . Equivalent are:

- (a)  $R$  is expressible in  $\Gamma$  (i.e.,  $R \in \langle \Gamma \rangle$ ).
- (b)  $I_k(\Gamma)$  is a gadget for expressing  $R$  with construction site  $(v_1, \dots, v_n)$ , where for each  $1 \leq i \leq n$ ,

$$v_i := (t_1[i], \dots, t_k[i]).$$

## Proof.

The direction from (b) to (a) is trivial, since  $I_k(\Gamma)$  is contained in  $\text{CSP}(\Gamma)$ . The other direction will be proved later.  $\square$

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# Example: $\neg, \oplus$

**Problem:** Is the implication expressible in the Boolean language  $\{\neg, \oplus\}$ ?

Consider the 3rd indicator problem (since  $R_{\Rightarrow}$  has three elements  $(1, 1), (0, 1), (0, 0)$ ). Consider the variables  $v = (1, 0, 0)$  and  $w = (1, 0, 1)$ :

1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

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1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

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1	1		1	1		0	0	0	0
1	1		0	0		1	1	0	0
1	0		1	0		1	0	1	0

Solutions

1	0		1	1		0	0	1	0
1	0		1	0		1	0	1	0
1	0		0	1		0	1	1	0
1	0		0	0		1	1	1	0
1	1		1	1		0	0	0	0
1	1		1	0		1	0	0	0
1	1		0	1		0	1	0	0
1	1		0	0		1	1	0	0

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1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

From this we obtain that  $\pi_{(v,w)}(I_3(\Gamma)) = D \times D \neq R_{\Rightarrow}$ .

Thus, the implication is not expressible.

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# Polymorphisms

Let  $f$  be a  $k$ -ary operation, i.e., a function  $f : D^k \rightarrow D$ .

For any collection of  $n$ -tuples,  $t_1, \dots, t_k \in D^n$ , let  $f(t_1, \dots, t_k)$  be defined as the  $n$ -tuple:

$$(f(t_1[1], \dots, t_k[1]), \dots, f(t_1[n], \dots, t_k[n])).$$

## Definition

Let  $f : D^k \rightarrow D$  be a  $k$ -ary operation, and  $R$  be an  $n$ -ary relation.

$f$  is a **polymorphism** of  $R$  (or:  $R$  is **invariant** under  $f$ ) if for all  $t_1, \dots, t_k \in R$ ,  $f(t_1, \dots, t_k) \in R$ .

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# Polymorphisms and Invariant Relations

Let  $\Gamma$  be a set of relations on a fixed domain  $D$ , and let  $F$  be a set of operations on  $D$ . Then define:

$\text{Pol}(\Gamma)$  : the set of operations on  $D$  that preserve each relation in  $\Gamma$

$\text{Inv}(F)$  : the set of relations on  $D$  that are invariant under each operation of  $F$

## Lemma

*Pol and Inv define anti-monotone functions, and are related by the following Galois connection:*

$$\Gamma \subseteq \text{Inv}(F) \iff F \subseteq \text{Pol}(\Gamma).$$

*In particular, it holds:*

$$\Gamma \subseteq \text{Inv}(\text{Pol}(\Gamma)) \quad \text{and} \quad F \subseteq \text{Pol}(\text{Inv}(F)).$$

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# The Indicator Problem and Polymorphisms

## Lemma

*Let  $\Gamma$  be a constraint language. The solutions of the  $k$ -th indicator problem  $I_k(\Gamma)$  are precisely the  $k$ -ary polymorphisms of  $\Gamma$ .*

## Proof.

Apply the definitions . . .

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# Expressiveness and Polymorphisms

## Lemma

*Let  $\Gamma$  be a constraint language over some domain  $D$ . If  $f : D^k \rightarrow D$  is a polymorphism of each  $R \in \Gamma$ , then  $f$  is a polymorphism of each  $R \in \langle \Gamma \rangle$ .*

## Proof.

Induction on primitive positive formula (cf. blackboard). □

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# Expressiveness and the Indicator Problem (Part 2)

The following lemma completes the proof of Jeavons' theorem:

## Lemma

Let  $R = \{t_1, \dots, t_k\}$  be an  $n$ -ary relation (over some finite domain  $D$ ). For  $1 \leq i \leq n$ , set  $v_i := (t_1[i], \dots, t_k[i])$ .  
If  $R$  is expressible in  $\Gamma$ , then  $R = \pi_{v_1, \dots, v_n}(\text{Sol}(I_k(\Gamma)))$ .

## Proof.

Blackboard. □

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# Expressiveness and Invariants

## Theorem

For any constraint language  $\Gamma$  over some finite domain  $D$ ,

$$\langle \Gamma \rangle = \text{Inv}(\text{Pol}(\Gamma))$$

## Proof.

$\subseteq$  is clear. For the converse let  $R$  be an  $n$ -ary relation that is invariant for each polymorphism of  $\Gamma$ . We have to show that  $R \in \langle \Gamma \rangle$ . Let  $R = \{t_1, \dots, t_k\}$  and consider the  $k$ -th indicator problem of  $\Gamma$ . First define  $v_i := (t_1[i], \dots, t_k[i])$  ( $1 \leq i \leq n$ ), then consider  $R_t = \pi_{v_1, \dots, v_n}(\text{Sol}(I_k(\Gamma)))$ . By one of the lemmas above,  $R$  is expressible if  $R = R_t$ .

$R_t \subseteq R$  follows from the facts that every solution of  $I_k(\Gamma)$  is a  $k$ -ary polymorphism and that each polymorphism of  $\Gamma$  preserves  $R$ . For  $R \subseteq R_t$ , consider  $t_j$  in  $R$ . Now the  $j$ -th projection function  $p_j : D^k \rightarrow D$  is a polymorphism. Hence  $t_j = p_j(t_1, \dots, t_k) \in R$ .  $\square$

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# Expressiveness, Polymorphisms, and Complexity

## Corollary

*A relation  $R$  on a finite domain is expressible by a constraint language if and only if  $\text{Pol}(\Gamma) \subseteq \text{Pol}(\{R\})$ .*

## Corollary

*Let  $\Gamma$  and  $\Delta$  be a constraint languages on a finite domain. If  $\Delta$  is finite and  $\text{Pol}(\Gamma) \subseteq \text{Pol}(\Delta)$ , then  $\text{CSP}(\Delta)$  is polynomial-time reducible to  $\text{CSP}(\Gamma)$ .*

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# Operations

Following, we study  $k$ -ary operations  $f : D^k \rightarrow D$ .

## Definition

- $f$  is **idempotent**, if for each  $x \in D$ ,  $f(x, \dots, x) = x$ .
- Given  $k = 3$ ,  $f$  is a **majority operation**, if for all  $x, y \in D$ ,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x.$$

- Given  $k = 3$ ,  $f$  is a **Mal'tsev operation**, if for all  $x, y \in D$ ,

$$f(y, y, x) = f(x, y, y) = x.$$

- $f$  is **conservative**, if for all  $x_1, \dots, x_k \in D$ ,

$$f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}.$$

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# Operations (cont'd)

## Definition

- Given  $k = 2$ ,  $f$  is a **semi-lattice operation**, if it is
  - associative (i.e.,  $f(x, f(y, z)) = f(f(x, y), z)$ ),
  - commutative (i.e.,  $f(x, y) = f(y, x)$ ), and
  - idempotent.
- Given  $k = 3$  and an Abelian group structure on  $D$ ,  $f$  is **affine**, if for all  $x, y, z \in D$ ,

$$f(x, y, z) = x - y + z.$$

- Given  $k \geq 3$ ,  $f$  is a **near-unanimity operation**, if for all  $x, y \in D$ ,

$$f(y, x, \dots, x) = f(x, y, x \dots, x) = \dots = f(x, \dots, x, y) = x.$$

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# Operations (cont'd)

## Definition

- $f$  is **essentially unary**, if there exists an  $1 \leq i \leq k$  and a unary non-constant operation  $g$  on  $D$  such that for all  $x_1, \dots, x_k \in D$ ,

$$f(x_1, \dots, x_k) = g(x_i).$$

If  $g$  is the identity operation, then  $f$  is called a **projection**.

- Given  $k \geq 3$ ,  $f$  is a **semi-projection** if  $f$  is not an projection and there exists an  $1 \leq i \leq k$ , such that for all  $x_1, \dots, x_k \in D$  with  $|\{x_1, \dots, x_k\}| < k$ ,

$$f(x_1, \dots, x_k) = x_i.$$

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# A Necessary Condition for Tractability

## Theorem

*Given  $P \neq NP$ , any tractable constraint language  $\Gamma$  over a finite domain has a solution to an indicator problem  $I_k(\Gamma)$  that defines*

- *a constant operation,*
- *a majority operation,*
- *an idempotent binary operation,*
- *an affine operation, or*
- *a semi-projection.*

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# Boolean CSPs

The complexity of any language over a domain of size 2 can be determined by considering the solutions of its 3rd order indicator problem. The problem is intractable unless this indicator problem has one of the following six solutions:

Variables								Schaefer class	Name
Solutions									
1	1	1	1	0	0	0	0	1	Constant 0
1	1	0	0	1	1	0	0	2	Constant 1
1	0	1	0	1	0	1	0	3	Anti-Horn
1	0	0	0	0	0	0	0	4	Horn-SAT
1	1	1	0	1	0	0	0	5	2-SAT
1	0	0	1	0	1	1	0	6	Linear

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Example:  $\neg, \oplus$

●	●	●	●	●	●	●	●
1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0
<hr/>							
Solutions							
<hr/>							
1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
<b>1</b>	<b>1</b>	<b>1</b>	<b>0</b>	<b>1</b>	<b>0</b>	<b>0</b>	<b>0</b>
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

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# Sufficient Conditions: Semi-Lattice Operations

In what follows let  $\Gamma$  be always be a constraint language over a finite domain  $D$ . We present some sufficient criteria for (in-)tractability.

## Theorem

*If  $\text{Pol}(\Gamma)$  contains a semi-lattice operation, then*

- *$\Gamma$  is tractable, and*
- *each instance of  $\text{CSP}(\Gamma)$  can be solved by enforcing generalized arc consistency.*

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# Examples

## Example 1:

If  $\Gamma$  is the Boolean constraint language containing all relations expressible by conjunctions of **Horn clauses**, then

$$\wedge : \{0, 1\}^2 \rightarrow \{0, 1\}$$

is a semi-lattice operation that is a polymorphism of  $\Gamma$ .

## Example 2:

If  $D$  is ordered, then  $\max$  is a semi-lattice operation, which is a polymorphism of each set of  $\max$ -closed relations.

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# Sufficient Conditions: Conservative Operations

## Theorem

*If  $\text{Pol}(\Gamma)$  contains a conservative and commutative operation, then  $\Gamma$  is tractable.*

Note: If  $\Gamma$  contains all unary relations on  $D$ , then all operations in  $\text{Pol}(\Gamma)$  are conservative.

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# Sufficient Conditions: Near-Unanimity Operations

## Theorem

*If  $\text{Pol}(\Gamma)$  contains a  $k$ -ary near-unanimity operation, then*

- *$\Gamma$  is tractable.*
- *Each instance of  $\text{CSP}(\Gamma)$  can be solved by enforcing strong  $k$ -consistency.*

## Proof.

Blackboard.

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# Examples

## Example 3:

Let  $\Gamma$  be the Boolean constraint language that consists of all relations definable by a PL-formula in CNF s. t. each conjunct has at most two literals.

Then

$$d(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

is a near-unanimity operation on  $\{0, 1\}$  and a polym. of  $\Gamma$ .

## Example 4:

The 0/1/all relations are invariant under the ternary operation

$$d(x, y, z) := \begin{cases} x & \text{if } y \neq z \\ y & \text{else} \end{cases}$$

which is a near-unanimity operation.

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# Sufficient Conditions: Mal'tsev Operations

## Theorem

*If  $\text{Pol}(\Gamma)$  contains a  $k$ -ary Mal'tsev operation, then  $\text{CSP}(\Gamma)$  is tractable.*

Note: Affine relations are Mal'tsev operations.

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# Reduced Constraint Languages

## Lemma

*Let  $\Gamma$  be a constraint language over  $D$ , and let  $f$  be a unary operation on  $\text{Pol}(\Gamma)$ . Let  $f(\Gamma)$  be the set of all  $f(R) := \{f(t) : t \in R\}$  with  $R \in \Gamma$ . Then,  $\text{CSP}(\Gamma)$  is polynomial-time equivalent to  $\text{CSP}(f(\Gamma))$ .*

## Definition

A constraint language  $\Gamma$  is **reduced** if all its unary polymorphisms are surjective.

Note: Each constraint language can be transformed into a reduced language. For this find all unary polymorphisms by generating and solving the 1st order indicator problem. Choose one of these polymorphisms  $f$  with a minimal number of values in its range.

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# A Sufficient Condition for Intractability

## Theorem

*Let  $\Gamma$  be a constraint language over a finite domain. If  $\text{Pol}(\Gamma)$  contains only essentially unary operations, then  $\text{CSP}(\Gamma)$  is NP-complete.*

## Proof idea:

We can assume that  $\Gamma$  is reduced. One can show that

- $\neq_D$  is in  $\text{Inv}(\text{Pol}(\Gamma))$ ;
- if  $|D| = 2$ ,  $\text{Inv}(\text{Pol}(\Gamma))$  contains the not-all-equal relation:

$$D^3 \setminus \{(x, x, x) : x \in D\}$$

which ensures that  $\text{CSP}(\Gamma)$  intractable. □

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# Towards a Classification

It can be shown that for any reduced constraint language  $\Gamma$  on a finite domain  $D$ , one of the following conditions holds:

- $\text{Pol}(\Gamma)$  contains a constant operation;
- $\text{Pol}(\Gamma)$  contains a ternary near-unanimity operation;
- $\text{Pol}(\Gamma)$  contains a Mal'tsev operation;
- $\text{Pol}(\Gamma)$  contains an idempotent binary operation;
- $\text{Pol}(\Gamma)$  contains a semi-projection;
- $\text{Pol}(\Gamma)$  contains essentially unary operations only.

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# Maximal and Maximal Tractable Languages

## Definition

- A constraint language  $\Gamma$  is **maximal tractable**, if it is tractable and for each relation  $R \notin \Gamma$ ,  $\Gamma \cup \{R\}$  is intractable.
- A constraint language  $\Gamma$  is **maximal**, if there is a relation  $R \notin \langle \Gamma \rangle$  and each proper extension of  $\langle \Gamma \rangle$  contains all relations on  $D$ .

Note: If  $\Gamma$  is a maximal language that is tractable, then  $\langle \Gamma \rangle$  is maximal tractable.

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# Maximality vs. Tractability

## Theorem

Let  $\Gamma$  be a constraint language on some finite domain  $D$ , and let  $f$  be a  $k$ -ary operation such that  $\langle \Gamma \rangle = \text{Inv}(\{f\})$ .

Then  $\langle \Gamma \rangle$  is maximal tractable, if

- $f$  is a constant operation;
- $f$  is a ternary near-unanimity operation;
- $f$  is a semi-lattice operation;
- $f$  is an affine operation.

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