

Constraint Satisfaction Problems

Tractable Constraint Languages

Bernhard Nebel and **Stefan Wöfl**

based on a slideset by
Malte Helmert and Stefan Wöfl
(summer term 2007)

Albert-Ludwigs-Universität Freiburg

November 23/25/30, 2009

Constraint Satisfaction Problems

November 23/25/30, 2009 — Tractable Constraint Languages

Tractable Constraint Languages

Schaefer's Dichotomy Theorem

Relational Clones

Expressiveness

Polymorphisms

Tractability over Finite Domains

Maximal Tractable Constraint Languages

Expressiveness vs. Complexity

- ▶ For some restricted constraint languages we know some polynomial time algorithms that solve each instance of that language
- ▶ Restricting constraint languages entails restricting expressiveness, i.e., the class of problems that can be expressed in the language
- ↪ How can we weight expressiveness against performance and vice versa?

CSP Instances aka Constraint Networks

Definition

An **instance of a constraint satisfaction problem** (i.e., a **constraint network**) is a triple

$$P = \langle V, D, C \rangle,$$

where:

- ▶ V is a non-empty and finite set of **variables**,
- ▶ D is an arbitrary set (**domain**),
- ▶ C is a finite set of **constraints** C_1, \dots, C_q , i.e., each constraint C_i is a pair (s_i, R_i) , where s_i is a tuple of variables of length m_i and R_i is an m_i -ary relation on D
(s_i : **constraint scope**; R_i : **constraint relation**).

Restricting the General CSP

The **general CSP decision problem** is the following: Given an instance of a constraint satisfaction problem, P , determine if there exists solution to P , i.e., determine whether

$$\text{Sol}(P) \\ := \{(d_1, \dots, d_n) \in D^n : a(v_i) = d_i \text{ for a solution } a \text{ of } P\}$$

(where n is the number of variables of V) is not empty.

Restricting the general CSP:

- ▶ **structural restriction:** consider just CSP instances with particular constraint scopes (e. g., where the network hypergraph has specific properties)
- ▶ **relational restriction:** consider just CSP instances, where the constraint relations have a specific form or specific properties

Constraint Language

Definition

A **constraint language** is an arbitrary set of relations, Γ , defined over some fixed domain (denoted by $D(\Gamma)$).

Definition

For a constraint language Γ , let $\text{CSP}(\Gamma)$ be the class of CSP instances $P = \langle V, D, C \rangle$ such that for each $(s, R) \in C$, $R \in \Gamma$. $\text{CSP}(\Gamma)$ is called the **relational subclass** associated with Γ .

Definition

A finite constraint language Γ is **tractable** if there exists a polynomial algorithm that solves all instances of $\text{CSP}(\Gamma)$.

An infinite constraint language Γ is **tractable** if each finite subset of the language is tractable.

Following, we present some examples:

Example: the CHIP language

CHIP is a constraint language for arithmetic and other constraints. **Basic constraints** in CHIP are so-called:

- ▶ **domain constraints**: unary constraints that restrict the domains of variables to a finite set of natural numbers
- ▶ **arithmetic constraints**: constraints of one of the forms

$$ax = by + c$$

$$ax \leq by + c$$

$$ax \geq by + c$$

($a, b, c \in \mathbb{N}, a \neq 0$). If these equations are conceived of as relations, the resulting constraint language is tractable.

The language is still tractable if we allow for relations expressed by

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq by + c$$

$$ax_1 \cdots x_n \geq by + c$$

$$(a_1x_1 \geq b_1) \vee \cdots \vee (a_nx_n \geq b_n) \vee (ay \geq b)$$

Example: Linear Relations

Let D be any field (e.g., the field of real numbers).

A **linear relation** on D is any relation defined by some system of linear equations:

$$a_1x_1 + \cdots + a_nx_n = r \quad (a_1, \dots, a_n, r \in D).$$

Then any instance of $\text{CSP}(\Gamma_{\text{lin}})$ can be represented by a system of linear equations over D , and hence be solved in polynomial time (apply Gaussian elimination).

Hence, the language of all linear relations over D is tractable.

Example: Relations on Ordered Finite Sets

Let D be an ordered and finite set.

Consider the binary **disequality relation**

$$\neq_D = \{(d_1, d_2) \in D^2 : d_1 \neq d_2\}$$

The class of CSP instances $\text{CSP}(\{\neq_D\})$ corresponds to the graph colorability problem with $|D|$ colors.

$\text{CSP}(\{\neq_D\})$ is tractable if $|D| \leq 2$, and intractable, otherwise.

The ternary **betweenness relation** over D is defined by:

$$B_D = \{(a, b, c) \in D^3 : a < b < c \vee c < b < a\}$$

$\text{CSP}(\{B_D\})$ is tractable if $|D| \leq 4$, and intractable if $|D| \geq 5$.

Example: Connected Row-Convex Relations

Let $D = \{d_1, \dots, d_n\}$ be an ordered and finite set.

For a binary relation R over D , the matrix representation of R is an $n \times n$ 0,1-matrix M , where $M_{ij} = 1$ iff $(d_i, d_j) \in R$.

The **pruned matrix representation** of R results from the matrix representation of R , when we remove all rows and columns in which only 0's occur.

R is **connected row-convex**, if in the pruned matrix representation of R , the pattern of 1's is connected along each column, along each row, and forms a connected 2-dimensional region.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The constraint language on any class of connected row-convex relations is tractable.

Example: Boolean Constraints

Let $D = \{d_0, d_1\}$.

The class of CSP instances $\text{CSP}(\{N_D\})$, where

$$N_D = D^3 \setminus \{(d_0, d_0, d_0), (d_1, d_1, d_1)\}$$

is the **not-all-equal relation** over D , is intractable.

$\text{CSP}(\{N_D\})$ corresponds to the not-all-equal satisfiability problem (NAE-3SAT), which is known to be NP-hard.

The class of CSP instances $\text{CSP}(\{T_D\})$, where

$$T_D = \{(d_0, d_0, d_1), (d_0, d_1, d_0), (d_1, d_0, d_0)\},$$

is intractable.

$\text{CSP}(\{T_D\})$ corresponds to the one-in-three satisfiability problem (1-in-3 SAT).

Example: 0/1/all-Relations

Let D be an arbitrary finite set. A relation R over D is called **0/1/all-relation** if one of the following conditions holds:

- ▶ R is unary;
- ▶ $R = D_1 \times D_2$ for subsets D_1, D_2 of D ;
- ▶ $R = \{(d, \pi(d)) : d \in D_1\}$, for some subset $D_1 \subseteq D$ and some permutation π of D ;
- ▶ $R = \{(a, b) \in D_1 \times D_2 : a = d_1 \vee b = d_2\}$, for some subsets D_1, D_2 of D and some elements $d_1 \in D_1, d_2 \in D_2$.

The language defined by all 0/1/all-relations is tractable.

It is even **maximal tractable**: if we add any binary relation over D that is not a 0/1/all-relation, then the resulting constraint language becomes intractable.

max-Closed Relations

Let $(D, <)$ be a linear order. Define $\max : D \times D \rightarrow D$ in the usual way, i.e., $\max(a, b) = a$ if $a > b$, and $\max(a, b) = b$, otherwise.

We extend \max to a function that can be applied to tuples, i.e., we define $\max : D^k \times D^k \rightarrow D^k$ by

$$\max((a_1, \dots, a_k), (b_1, \dots, b_k)) \\ := (\max(a_1, b_1), \dots, \max(a_k, b_k)).$$

Definition

An n -ary relation R over D is **max-closed** if for all $(a_1, \dots, a_n), (b_1, \dots, b_n) \in R$,

$$\max((a_1, \dots, a_n), (b_1, \dots, b_n)) \in R.$$

max-Closed Relations and Tractability

Lemma

Let Γ be a constraint language with max-closed relations only. Then $CSP(\Gamma)$ is tractable.

Proof.

Enforce generalized arc consistency. If any domain of the resulting network is empty, the network is inconsistent. Otherwise, set each variable to its maximal value, □

Example: max-Closed Relations

Consider the CHIP language. All relations of CHIP are max-closed. Hence any set of equations can be solved by establishing arc consistency.

For example, consider a CSP instance with domain $\{1, \dots, 5\}$, variables $\{v, w, x, y, z\}$, and equations

$$w \neq 3, \quad z \neq 5, \quad 3v \leq z, \quad y \geq z + 2,$$

$$3x + y + z \geq 5w + 1, \quad wz \geq 2y.$$

Enforcing arc consistency results in:

$$D(v) = \{1\}, \quad D(w) = \{4\}, \quad D(x) = \{3, 4, 5\},$$

$$D(y) = \{5\}, \quad D(z) = \{3\}.$$

Hence

$$v \mapsto 1, \quad w \mapsto 4, \quad x \mapsto 5, \quad y \mapsto 5, \quad z \mapsto 3$$

is a solution of the constraint network.

Boolean Constraint Languages

The key result in the literature on tractable constraint languages is Schaefer's Dichotomy Theorem (1978).

Definition

A **Boolean constraint language** is a constraint language over the two-element domain $D = \{0, 1\}$.

Schaefer's theorem states that any Boolean constraint language is either tractable or NP-complete. Moreover, it provides a classification of all tractable constraint languages.

Definition

An arbitrary constraint language Γ is **NP-complete** if $\text{CSP}(\Delta)$ is NP-complete for some finite subset $\Delta \subseteq \Gamma$.

Schaefer's Theorem

Theorem (Schaefer 1978)

Let Γ be a Boolean constraint language. Then Γ is tractable if at least one of the following conditions is satisfied:

- 1. Each relation in Γ contains the tuple $(0, \dots, 0)$.*
- 2. Each relation in Γ contains the tuple $(1, \dots, 1)$.*
- 3. Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one negative literal.*
- 4. Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most one positive literal.*
- 5. Each relation in Γ is definable by a formula in CNF s. t. each conjunct has at most two literals.*
- 6. Each relation in Γ is the set of solutions of a system of linear equations over the finite field with 2 elements.*

In all other cases, Γ is NP-complete.

Algorithm Selector

Let Γ be a Boolean constraint language.

Class 1: any CSP instance P can be solved by simply assigning 0 to each variable of P .

Class 2: cf. Class 1 ($v \mapsto 1$).

Class 6: any CSP instance P can be solved by applying the Gaussian elimination procedure.

Class 5: any CSP instance P can be solved by resolution: in this case $\text{CSP}(\Gamma)$ corresponds to the 2-SAT satisfiability problem and this can be solved efficiently by resolution.

Class 4: any CSP instance P can be solved by unit resolution: here $\text{CSP}(\Gamma)$ corresponds to the Horn-SAT satisfiability problem, which can be solved efficiently by unit resolution.

Class 3: cf. Class 4 (“anti-Horn”).

Gadgets

Definition

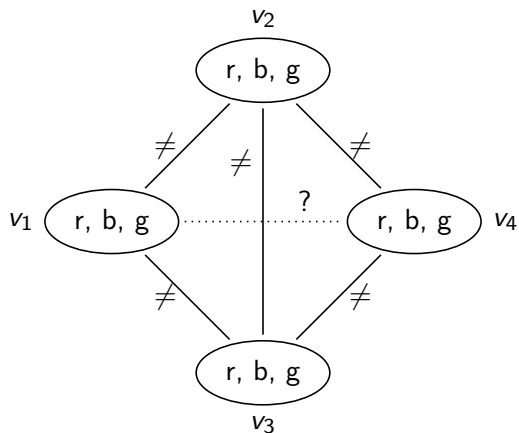
Let Γ be constraint language and R be a relation on $\Gamma(D)$.

R is **expressible** in Γ if there exists a CSP instance $P \in \text{CSP}(\Gamma)$ and a sequence of variables v_1, \dots, v_n such that

$$R = \pi_{v_1, \dots, v_n}(\text{Sol}(P)).$$

P is referred to as a **gadget** for expressing R in $\text{CSP}(\Gamma)$, the sequence v_1, \dots, v_n as **construction site** for R .

Example



Which relation is expressed by the edge (v_1, v_4) ?

Relational Clones

Expressiveness can also be reformulated in the following way:
Let Γ, Γ' be constraint languages (def. on the same domain D).

Definition

Γ' is a **relational clone** of Γ if Γ' contains each relation expressible by a FO-formula with

- ▶ relations from $\Gamma \cup \{=_{D}\}$,
- ▶ conjunctions, and
- ▶ existential quantification.

(Formulae of this form are called **primitive positive formulae**.)

Definition

Let Γ be a constraint language. $\langle \Gamma \rangle$ denotes the smallest relational clone containing Γ , **the clone generated by Γ** .

Example

Consider a Boolean constraint language with the following relations:

$$R_1 = \{(0, 1), (1, 0), (1, 1)\} \quad R_2 = \{(0, 0), (0, 1), (1, 0)\}.$$

The relational clone generated by the set of these two relations contains all 16 binary Boolean relations. For example:

$R_3 := \{(0, 1), (1, 0)\}$	$R_1(v_1, v_2) \wedge R_2(v_1, v_2)$
$R_4 := \{(0, 0), (1, 0), (1, 1)\}$	$\exists y(R_1(v_1, y) \wedge R_2(y, v_2))$
$R_5 := \{(0, 0), (1, 1)\}$	$v_1 = v_2$
$R_6 := \{(0, 0)\}$	$R_2(v_1, v_2) \wedge R_5(v_1, v_2)$
$R_7 := \{(1, 1)\}$	$R_1(v_1, v_2) \wedge R_5(v_1, v_2)$
$R_8 := \{(0, 1)\}$	$\exists y(R_6(v_1, y) \wedge R_1(y, v_2))$

...

Reducibility I

Theorem

Let Γ be a set of relations on a fixed domain D , and let Δ be a finite subset of $\langle \Gamma \rangle$. Then there exists a polynomial time reduction from $CSP(\Delta)$ to $CSP(\Gamma)$.

Proof.

Let $\Delta = \{S_1, \dots, S_k\}$ be a finite set of relations, where each S_j is expressible by a pp-formula with relations from Γ and the relation $=_D$. For each S_j fix such a formula $\phi_j(x_1, \dots, x_{r_j})$, where r_j is the arity of S_j . Without loss of generality, we may assume that each $\phi_j(x_1, \dots, x_{r_j})$ has the form

$$\exists u_1 \dots u_m (R_1(w_1^1, \dots, w_{k_1}^1) \wedge \dots \wedge R_n(w_1^n, \dots, w_{k_n}^n)) \quad (1)$$

where $w_1^1, \dots, w_{k_1}^1, \dots, w_1^n, \dots, w_{k_n}^n \in \{x_1, \dots, x_{r_j}, u_1, \dots, u_m\}$ for some auxiliary variables u_1, \dots, u_m , and $R_1, \dots, R_n \in \Gamma \cup \{=_D\}$

...

Reducibility II

Let $P = \langle V, D, C \rangle$ be an arbitrary instance in $\text{CSP}(\Delta)$. Initially, set $V' := V, D' := D, C' := C$. For each constraint (s, R) (where $s = (v_1, \dots, v_r)$) of P , proceed as follows:

1. add the auxiliary variables u_1, \dots, u_m to V' (always add new variables, rename variables if necessary (also in (1)))
2. remove (s, R) from C' and instead add to C' the constraints (cf. (1)):

$$((w_1^1, \dots, w_{k_1}^1), R_1), \dots, (w_1^n, \dots, w_{k_n}^n, R_n)$$

The CSP instance P' obtained by this procedure is contained in $\text{CSP}(\Gamma \cup \{=_{D'}\})$ and is obviously equivalent to P . Furthermore, from P' we can obtain a CSP instance P'' in $\text{CSP}(\Gamma)$ by deleting constraints of the form $((v_i, v_j), =_{D'})$ and replacing any occurrence of v_j by v_i . Obviously, both transformation can be done in polynomial time. □

Reducibility III

Corollary

A constraint language Γ is tractable if and only if its relational clone $\langle \Gamma \rangle$ is tractable. Γ is NP-complete if and only if $\langle \Gamma \rangle$ is NP-complete.

Corollary

Let Γ be a constraint language and let R be a relation. R is expressible in Γ if and only if $R \in \langle \Gamma \rangle$.

The Indicator Problem

Let $k \geq 1$ be a fixed natural number.

Let $s = (x_1, \dots, x_m)$ be a list of k -tuples in D^k .

Let R be an n -ary relation on D .

We say, that s **matches** R if $n = m$ and if for each $1 \leq i \leq k$, the n -tuple $(x_1[i], \dots, x_n[i])$ is in R .

Let now Γ be a fixed constraint language. Set $I_k(\Gamma) = \langle V, D, C \rangle$, where

$$V := D^k$$

$$C := \{(s, R) : s \text{ matches } R\}$$

Note: $I_k(\Gamma) \in \text{CSP}(\Gamma)$ and contains constraints from Γ on every possible scope which matches some relation in Γ .

Definition

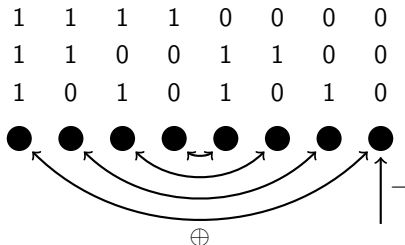
$I_k(\Gamma)$ is said to be the **indicator problem of order k** for Γ .

Example: \neg, \oplus

Consider the Boolean constraint language containing the unary relation \neg and the exclusive-or relation \oplus , i.e.,

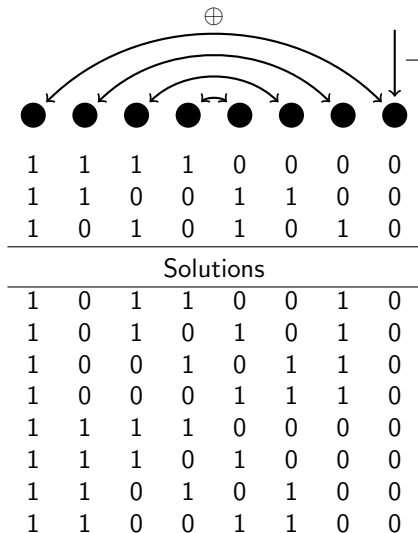
$$R_{\oplus} = \{(0, 1), (1, 0)\} \quad \text{and} \quad R_{\neg} = \{(0)\}.$$

The 3-rd order indicator problem of this language is:



Example (cont'd): \neg, \oplus

Solutions of this indicator problem:



Expressiveness and the Indicator Problem

Theorem (Jeavons (1998))

Let Γ be a constraint language over some finite domain D and let $R = \{t_1, \dots, t_k\}$ be any n -ary relation on D . Equivalent are:

- (a) R is expressible in Γ (i.e., $R \in \langle \Gamma \rangle$).
- (b) $I_k(\Gamma)$ is a gadget for expressing R with construction site (v_1, \dots, v_n) , where for each $1 \leq i \leq n$,

$$v_i := (t_1[i], \dots, t_k[i]).$$

Proof.

The direction from (b) to (a) is trivial, since $I_k(\Gamma)$ is contained in $\text{CSP}(\Gamma)$. The other direction will be proved later. □

Example: \neg, \oplus

Problem: Is the implication expressible in the Boolean language $\{\neg, \oplus\}$?

Consider the 3rd indicator problem (since R_{\Rightarrow} has three elements $(1, 1), (0, 1), (0, 0)$). Consider the variables $v = (1, 0, 0)$ and $w = (1, 0, 1)$:

1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

From this we obtain that $\pi_{(v,w)}(I_3(\Gamma)) = D \times D \neq R_{\Rightarrow}$.

Thus, the implication is not expressible.

Polymorphisms

Let f be a k -ary operation, i.e., a function $f : D^k \rightarrow D$.

For any collection of n -tuples, $t_1, \dots, t_k \in D^n$, let $f(t_1, \dots, t_k)$ be defined as the n -tuple:

$$(f(t_1[1], \dots, t_k[1]), \dots, f(t_1[n], \dots, t_k[n])).$$

Definition

Let $f : D^k \rightarrow D$ be a k -ary operation, and R be an n -ary relation.

f is a **polymorphism** of R (or: R is **invariant** under f) if for all $t_1, \dots, t_k \in R$, $f(t_1, \dots, t_k) \in R$.

Polymorphisms and Invariant Relations

Let Γ be a set of relations on a fixed domain D , and let F be a set of operations on D . Then define:

$\text{Pol}(\Gamma)$: the set of operations on D that preserve each relation in Γ

$\text{Inv}(F)$: the set of relations on D that are invariant under each operation of F

Lemma

Pol and Inv define anti-monotone functions, and are related by the following Galois connection:

$$\Gamma \subseteq \text{Inv}(F) \iff F \subseteq \text{Pol}(\Gamma).$$

In particular, it holds:

$$\Gamma \subseteq \text{Inv}(\text{Pol}(\Gamma)) \quad \text{and} \quad F \subseteq \text{Pol}(\text{Inv}(F)).$$

The Indicator Problem and Polymorphisms

Lemma

Let Γ be a constraint language. The solutions of the k -th indicator problem $I_k(\Gamma)$ are precisely the k -ary polymorphisms of Γ .

Proof.

Apply the definitions . . .



Expressiveness and Polymorphisms

Lemma

Let Γ be a constraint language over some domain D . If $f : D^k \rightarrow D$ is a polymorphism of each $R \in \Gamma$, then f is a polymorphism of each $R \in \langle \Gamma \rangle$.

Proof.

Induction on primitive positive formula (cf. blackboard). □

Expressiveness and the Indicator Problem (Part 2)

The following lemma completes the proof of Jeavons' theorem:

Lemma

Let $R = \{t_1, \dots, t_k\}$ be an n -ary relation (over some finite domain D).

For $1 \leq i \leq n$, set $v_i := (t_1[i], \dots, t_k[i])$.

If R is expressible in Γ , then $R = \pi_{v_1, \dots, v_n}(\text{Sol}(I_k(\Gamma)))$.

Proof.

Blackboard.



Expressiveness and Invariants

Theorem

For any constraint language Γ over some finite domain D ,

$$\langle \Gamma \rangle = \text{Inv}(\text{Pol}(\Gamma))$$

Proof.

\subseteq is clear. For the converse let R be an n -ary relation that is invariant for each polymorphism of Γ . We have to show that $R \in \langle \Gamma \rangle$. Let $R = \{t_1, \dots, t_k\}$ and consider the k -th indicator problem of Γ . First define $v_i := (t_1[i], \dots, t_k[i])$ ($1 \leq i \leq n$), then consider $R_t = \pi_{v_1, \dots, v_n}(\text{Sol}(I_k(\Gamma)))$. By one of the lemmas above, R is expressible if $R = R_t$.

$R_t \subseteq R$ follows from the facts that every solution of $I_k(\Gamma)$ is a k -ary polymorphism and that each polymorphism of Γ preserves R . For $R \subseteq R_t$, consider t_j in R . Now the j -th projection function $p_j : D^k \rightarrow D$ is a polymorphism. Hence $t_j = p_j(t_1, \dots, t_k) \in R$. □

Expressiveness, Polymorphisms, and Complexity

Corollary

A relation R on a finite domain is expressible by a constraint language if and only if $\text{Pol}(\Gamma) \subseteq \text{Pol}(\{R\})$.

Corollary

Let Γ and Δ be a constraint languages on a finite domain. If Δ is finite and $\text{Pol}(\Gamma) \subseteq \text{Pol}(\Delta)$, then $\text{CSP}(\Delta)$ is polynomial-time reducible to $\text{CSP}(\Gamma)$.

Operations

Following, we study k -ary operations $f : D^k \rightarrow D$.

Definition

- ▶ f is **idempotent**, if for each $x \in D$, $f(x, \dots, x) = x$.
- ▶ Given $k = 3$, f is a **majority operation**, if for all $x, y \in D$,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x.$$

- ▶ Given $k = 3$, f is a **Mal'tsev operation**, if for all $x, y \in D$,

$$f(y, y, x) = f(x, y, y) = x.$$

- ▶ f is **conservative**, if for all $x_1, \dots, x_k \in D$,

$$f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}.$$

Operations (cont'd)

Definition

- ▶ Given $k = 2$, f is a **semi-lattice operation**, if it is
 - ▶ associative (i.e., $f(x, f(y, z)) = f(f(x, y), z)$),
 - ▶ commutative (i.e., $f(x, y) = f(y, x)$), and
 - ▶ idempotent.
- ▶ Given $k = 3$ and an Abelian group structure on D , f is **affine**, if for all $x, y, z \in D$,

$$f(x, y, z) = x - y + z.$$

- ▶ Given $k \geq 3$, f is a **near-unanimity operation**, if for all $x, y \in D$,

$$f(y, x, \dots, x) = f(x, y, x, \dots, x) = \dots = f(x, \dots, x, y) = x.$$

Operations (cont'd)

Definition

- ▶ f is **essentially unary**, if there exists an $1 \leq i \leq k$ and a unary non-constant operation g on D such that for all $x_1, \dots, x_k \in D$,

$$f(x_1, \dots, x_k) = g(x_i).$$

If g is the identity operation, then f is called a **projection**.

- ▶ Given $k \geq 3$, f is a **semi-projection** if f is not an projection and there exists an $1 \leq i \leq k$, such that for all $x_1, \dots, x_k \in D$ with $|\{x_1, \dots, x_k\}| < k$,

$$f(x_1, \dots, x_k) = x_i.$$

A Necessary Condition for Tractability

Theorem

Given $P \neq NP$, any tractable constraint language Γ over a finite domain has a solution to an indicator problem $I_k(\Gamma)$ that defines

- ▶ *a constant operation,*
- ▶ *a majority operation,*
- ▶ *an idempotent binary operation,*
- ▶ *an affine operation, or*
- ▶ *a semi-projection.*

Boolean CSPs

The complexity of any language over a domain of size 2 can be determined by considering the solutions of its 3rd order indicator problem. The problem is intractable unless this indicator problem has one of the following six solutions:

Variables									
1	1	1	1	0	0	0	0		
1	1	0	0	1	1	0	0		
1	0	1	0	1	0	1	0		
Solutions								Schaefer class	Name
0	0	0	0	0	0	0	0	1	Constant 0
1	1	1	1	1	1	1	1	2	Constant 1
1	1	1	1	1	1	1	0	3	Anti-Horn
1	0	0	0	0	0	0	0	4	Horn-SAT
1	1	1	0	1	0	0	0	5	2-SAT
1	0	0	1	0	1	1	0	6	Linear

Example: \neg, \oplus

●	●	●	●	●	●	●	●
1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0

Solutions

1	0	1	1	0	0	1	0
1	0	1	0	1	0	1	0
1	0	0	1	0	1	1	0
1	0	0	0	1	1	1	0
1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	1	0	0	1	1	0	0

Sufficient Conditions: Semi-Lattice Operations

In what follows let Γ be always be a constraint language over a finite domain D . We present some sufficient criteria for (in-) tractability.

Theorem

If $\text{Pol}(\Gamma)$ contains a semi-lattice operation, then

- ▶ *Γ is tractable, and*
- ▶ *each instance of $\text{CSP}(\Gamma)$ can be solved by enforcing generalized arc consistency.*

Examples

Example 1:

If Γ is the Boolean constraint language containing all relations expressible by conjunctions of **Horn clauses**, then

$$\wedge : \{0, 1\}^2 \rightarrow \{0, 1\}$$

is a semi-lattice operation that is a polymorphism of Γ .

Example 2:

If D is ordered, then \max is a semi-lattice operation, which is a polymorphism of each set of \max -closed relations.

Sufficient Conditions: Conservative Operations

Theorem

If $\text{Pol}(\Gamma)$ contains a conservative and commutative operation, then Γ is tractable.

Note: If Γ contains all unary relations on D , then all operations in $\text{Pol}(\Gamma)$ are conservative.

Sufficient Conditions: Near-Unanimity Operations

Theorem

If $\text{Pol}(\Gamma)$ contains a k -ary near-unanimity operation, then

- ▶ *Γ is tractable.*
- ▶ *Each instance of $\text{CSP}(\Gamma)$ can be solved by enforcing strong k -consistency.*

Proof.

Blackboard. □

Examples

Example 3:

Let Γ be the Boolean constraint language that consists of all relations definable by a PL-formula in CNF s. t. each conjunct has at most two literals.

Then

$$d(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$$

is a near-unanimity operation on $\{0, 1\}$ and a polym. of Γ .

Example 4:

The 0/1/all relations are invariant under the ternary operation

$$d(x, y, z) := \begin{cases} x & \text{if } y \neq z \\ y & \text{else} \end{cases}$$

which is a near-unanimity operation.

Sufficient Conditions: Mal'tsev Operations

Theorem

If $\text{Pol}(\Gamma)$ contains a k -ary Mal'tsev operation, then $\text{CSP}(\Gamma)$ is tractable.

Note: Affine relations are Mal'tsev operations.

Reduced Constraint Languages

Lemma

Let Γ be a constraint language over D , and let f be a unary operation on $\text{Pol}(\Gamma)$. Let $f(\Gamma)$ be the set of all $f(R) := \{f(t) : t \in R\}$ with $R \in \Gamma$. Then, $\text{CSP}(\Gamma)$ is polynomial-time equivalent to $\text{CSP}(f(\Gamma))$.

Definition

A constraint language Γ is **reduced** if all its unary polymorphisms are surjective.

Note: Each constraint language can be transformed into a reduced language. For this find all unary polymorphisms by generating and solving the 1st order indicator problem. Choose one of these polymorphisms f with a minimal number of values in its range.

A Sufficient Condition for Intractability

Theorem

Let Γ be a constraint language over a finite domain. If $\text{Pol}(\Gamma)$ contains only essentially unary operations, then $\text{CSP}(\Gamma)$ is NP-complete.

Proof idea:

We can assume that Γ is reduced. One can show that

- ▶ \neq_D is in $\text{Inv}(\text{Pol}(\Gamma))$;
- ▶ if $|D| = 2$, $\text{Inv}(\text{Pol}(\Gamma))$ contains the not-all-equal relation:

$$D^3 \setminus \{(x, x, x) : x \in D\}$$

which ensures that $\text{CSP}(\Gamma)$ intractable. □

Towards a Classification

It can be shown that for any reduced constraint language Γ on a finite domain D , one of the following conditions holds:

- ▶ $\text{Pol}(\Gamma)$ contains a constant operation;
- ▶ $\text{Pol}(\Gamma)$ contains a ternary near-unanimity operation;
- ▶ $\text{Pol}(\Gamma)$ contains a Mal'tsev operation;
- ▶ $\text{Pol}(\Gamma)$ contains an idempotent binary operation;
- ▶ $\text{Pol}(\Gamma)$ contains a semi-projection;
- ▶ $\text{Pol}(\Gamma)$ contains essentially unary operations only.

Maximal and Maximal Tractable Languages

Definition

- ▶ A constraint language Γ is **maximal tractable**, if it is tractable and for each relation $R \notin \Gamma$, $\Gamma \cup \{R\}$ is intractable.
- ▶ A constraint language Γ is **maximal**, if there is a relation $R \notin \langle \Gamma \rangle$ and each proper extension of $\langle \Gamma \rangle$ contains all relations on D .

Note: If Γ is a maximal language that is tractable, then $\langle \Gamma \rangle$ is maximal tractable.

Maximality vs. Tractability





Theorem

Let Γ be a constraint language on some finite domain D , and let f be a k -ary operation such that $\langle \Gamma \rangle = \text{Inv}(\{f\})$.

Then $\langle \Gamma \rangle$ is maximal tractable, if

- ▶ *f is a constant operation;*
- ▶ *f is a ternary near-unanimity operation;*
- ▶ *f is a semi-lattice operation;*
- ▶ *f is an affine operation.*

Literature

-  David Cohen and Peter Jeavons.
Tractable constraint languages.
In: R. Dechter *Constraint Processing*, Chapter 11, Morgan Kaufmann, 2003
-  Andrei Bulatov, Andrei Krokhin, and Peter Jeavons.
The complexity of maximal constraint languages.
In: *Proceedings of STOC'01*, 2001
-  Andrei Bulatov, P. Jeavons, and Andrei Krokhin.
Classifying the complexity of constraint using finite algebras.
SIAM J. Comput. 34(3), 2005
-  David Cohen and Peter Jeavons.
The complexity of constraint languages.
In: F. Rossi, P. v. Beek, and T. Walsh, *Handbook of Constraint Programming*, Elsevier, 2006