Constraint Satisfaction Problems

Enforcing Consistency

Bernhard Nebel and **Stefan Wölfl**

based on a slideset by

Malte Helmert and Stefan Wölfl

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Enforcing Consistency

- The more explicit and tight constraint networks are, the more restricted is the search space of partial solutions.
- Idea: infer at least a limited number of new constraints (by methods called local consistency-enforcing, bounded consistency inference, constraint propagation).
- Consistency-enforcing algorithms aim at *assisting search*: How can we extend a given partial solution of a small subnetwork to a partial solution of a larger subnetwork?

Convention

In what follows we will always assume that the variables of a constraint network appear in some order. Then we can write constraint networks in the form:

$$\mathcal{C} = \langle V, D, C \rangle ,$$

where D_i is the (possibly empty) domain of variable v_i, and constraints in the form R_{ijk}, where $\{v_i, v_j, v_k\}$ is the scope of the relation.

Further, we assume that C does not contain unary constraints, i.e., constraints in C are always relations with arity $n > 1$.

This is possible, since we can define:

$$D_i := \mathrm{dom}(v_i) \cap R_{v_i}$$

and then delete R_{v_i} from the original network.

D_i will be referred to as domains, unary constraint, or domain constraint.

Arc Consistency

Let $\mathcal{C} = \langle V, D, C \rangle$ be a constraint network.

Definition

(a) A variable v_i is arc-consistent relative to variable v_j if for every value $a_i \in D_i$ there exists an $a_j \in D_j$ with $(a_i, a_j) \in R_{ij}$ (in case that R_{ij} exists in C).

(b) An "arc constraint" R_{ij} is arc-consistent if v_i is arc-consistent relative to v_j and v_j is arc-consistent relative to v_i.

(c) A network $\mathcal{C}$ is arc-consistent if all its arc constraints are arc-consistent.

Lemma

Checking whether a network $\mathcal{C} = \langle V, D, C \rangle$ is arc-consistent requires $e \cdot k^2$ operations (where e is the number of its binary constraints and k is an upper bound of its domain sizes).

Example

Consider a constraint network with two variables v_1 and v_2, domains $D_1 = D_2 = \{1, 2, 3\}$, and the binary constraint expressed by $v_1 < v_2$.

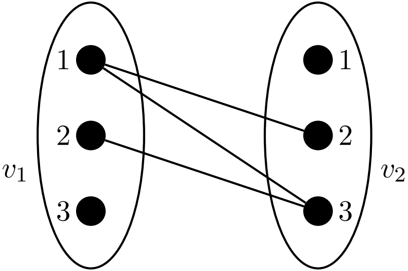

Figure: A network that is not arc-consistent

Revising a Single Domains

Revise (v_i, v_j):

Input: a network with two variables v_i, v_j, domains D_i and D_j, and constraint R_{ij}

Output: a network with refined D_i such that v_i is arc-consistent relative to v_j

for each $a_i \in D_i$
    if there is no $a_j \in D_j$ with $(a_i, a_j) \in R_{ij}$
        then delete a_i from D_i
    endif
endfor

This is equivalent to applying:

$$D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$$

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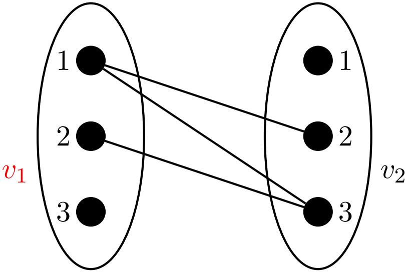
Revising a Single Domain

Lemma

The complexity of Revise is $\mathcal{O}(k^2)$, where k is an upper bound of the domain sizes.

Note: With a simple modification of the Revise algorithm one could improve to $\mathcal{O}(t)$, where t is the maximal number of tuples occurring in one of the binary constraints in the network.



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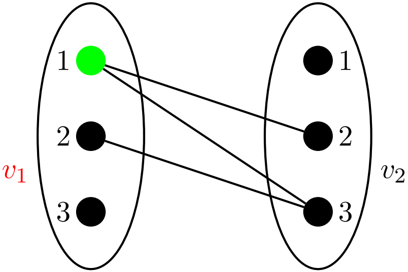

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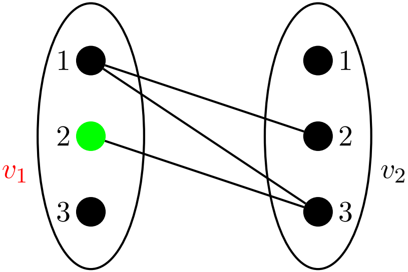

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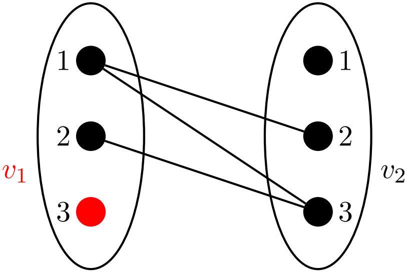

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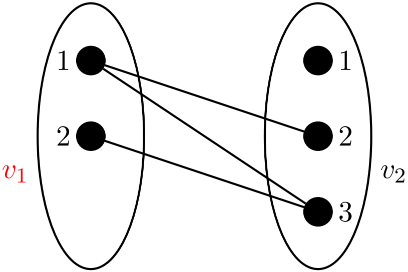


Enforcing Arc Consistency: AC-1

AC-1$(\mathcal{C})$:

Input: a constraint network $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but arc-consistent network $\mathcal{C}'$

repeat
    for each arc $\{v_i, v_j\}$ with $R_{ij} \in C$
        Revise(v_i, v_j)
        Revise(v_j, v_i)
    endfor
until no domain is changed

Enforcing Arc Consistency: AC-1

Lemma

Let $\mathcal{C}$ be a constraint network with n variables, each with a domain of size $\leq k$, and e binary constraints.
Applying AC-1 on the network runs in time $\mathcal{O}(e \cdot n \cdot k^3)$.

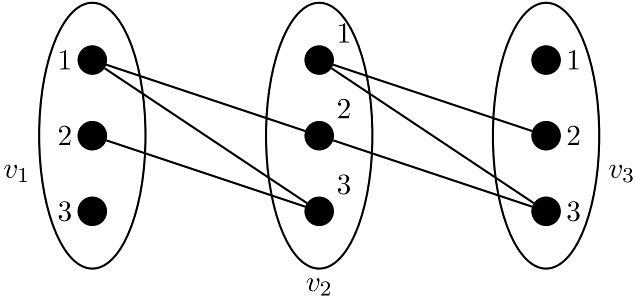
Proof.

One cycle through all binary constraints takes $\mathcal{O}(e \cdot k^2)$. In the worst case, one cycle just removes one value from one domain. Moreover, there are at most $n \cdot k$ values. This result in an upper bound of $\mathcal{O}(e \cdot n \cdot k^3)$. □

Note: If the input network is already arc-consistent, then AC-1 runs in time $\mathcal{O}(e \cdot k^2)$.

Example: AC-1

Consider a constraint network with three variables v_1, v_2, and v_3, domains $D_1 = D_2 = \{1, 2, 3\}$, and the binary constraints expressed by $v_1 < v_2$ and $v_2 < v_3$.



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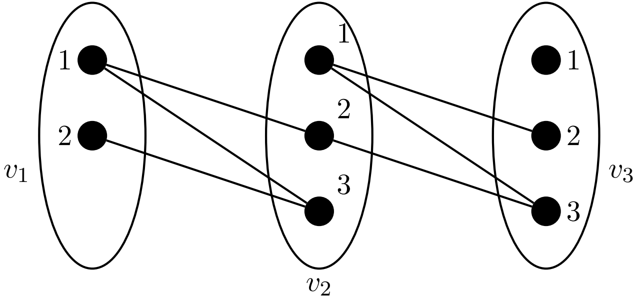

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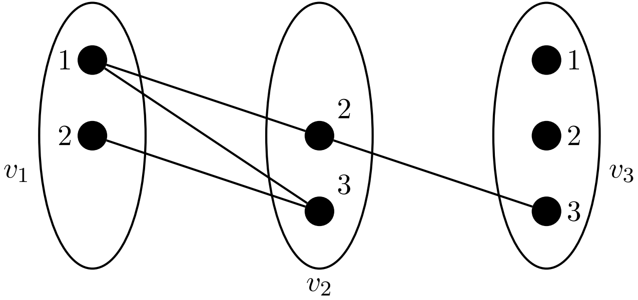

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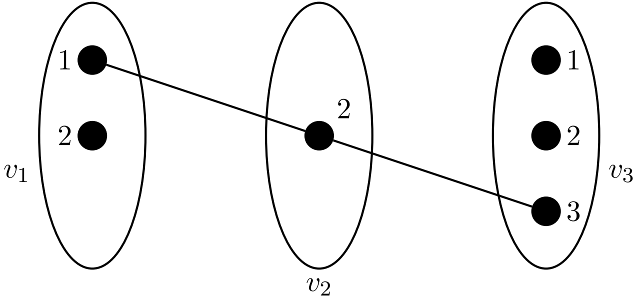

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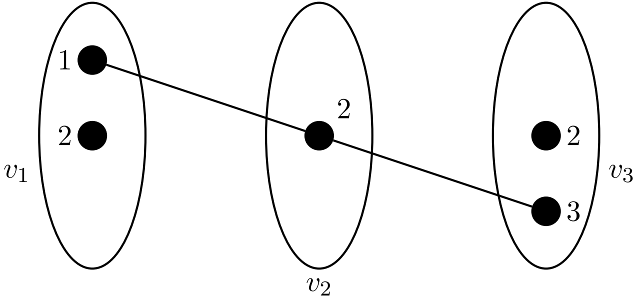

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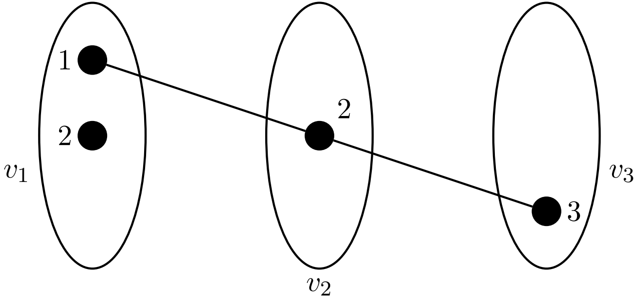

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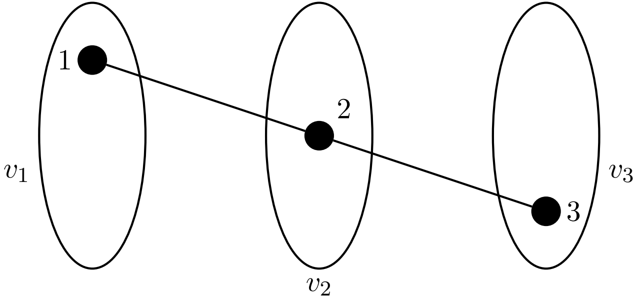

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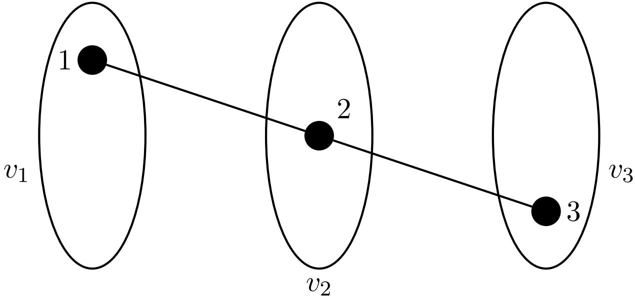


Note: Enforcing arc consistency may already be sufficient to show that a constraint network is inconsistent. For example, add the constraint $v_3 < v_1$ to the network just considered.

Enforcing Arc Consistency: AC-3

Idea: no need to process all constraints if only a few domains have changed. Operate on a queue of constraints to be processed.

AC-3($\mathcal{C}$):

Input: a constraint network $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but arc-consistent network $\mathcal{C}'$

for each pair v_i, v_j that occurs in a constraint R_{ij}
    queue $\leftarrow$ *queue* $\cup \{(v_i, v_j), (v_j, v_i)\}$
endfor
while *queue* is not empty
    select and delete (v_i, v_j) from *queue*
    Revise(v_i, v_j)
    if Revise(v_i, v_j) changes D_i
        then *queue* $\leftarrow$ *queue* $\cup \{(v_k, v_i) : k \neq i, k \neq j\}$
    endif
endwhile

Enforcing Arc Consistency: AC-3

Lemma

Let $\mathcal{C}$ be a constraint network with n variables, each with a domain of size $\leq k$, and e binary constraints.
Applying AC-3 on the network runs in time $\mathcal{O}(e \cdot k^3)$.

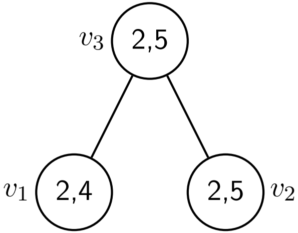
Proof.

Consider a single constraint. Each time, when it is reintroduced into the queue, the domain of one of its variables must have been changed. Since there are at most $2 \cdot k$ values, AC-3 processes each constraint at most $2 \cdot k$ times. Because we have e constraints and processing of each is in time $\mathcal{O}(k^2)$, we obtain $\mathcal{O}(e \cdot k^3)$. □

Note: If the input network is arc-consistent, then AC-3 runs in time $\mathcal{O}(e \cdot k^2)$.

Enforcing Arc Consistency: AC-3

Example: Consider a constraint network with 3 variables v_1, v_2, v_3 with domains $D_1 = \{2, 4\}$ and $D_2 = D_3 = \{2, 5\}$, and two constraints expressed by $v_3|v_1$ and $v_3|v_2$ ("divides").



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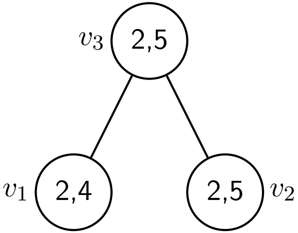

Queue
(v_1, v_3)

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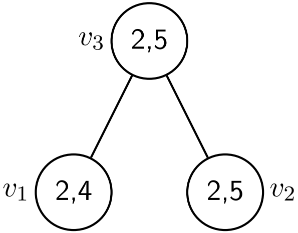

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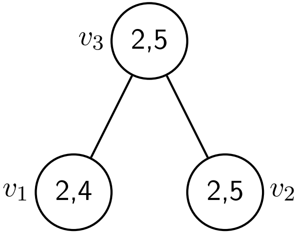

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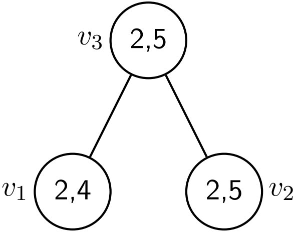

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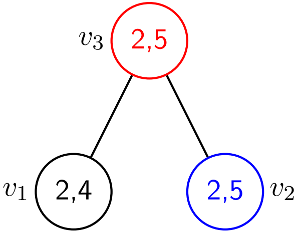

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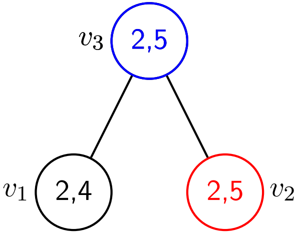

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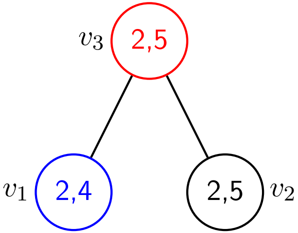

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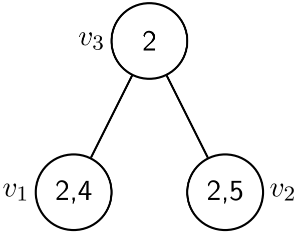

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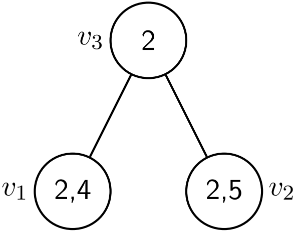

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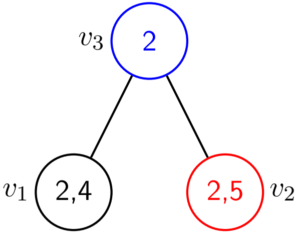

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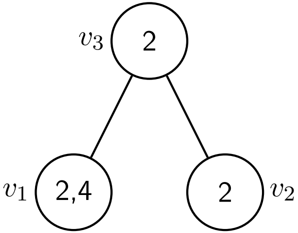

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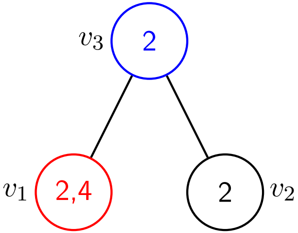

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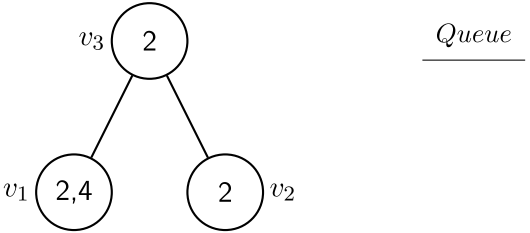


Enforcing Arc Consistency: AC-4

- To verify that a network is arc-consistent needs $e \cdot k^2$ operations.
- The following algorithm AC-4 achieves optimal performance, ...
- at the cost of "best case performance", which is $\Omega(e \cdot k^2)$.

Idea:

- Associate to each value a_i in the domain of variable v_i the amount of support from variable v_j (i.e., the number of values in D_j that are consistent with a_i);
- Delete a value a_i if it has no support from any other variable

Details:

- *List*: currently unsupported variable-value pairs;
- *counter*(x_i, a_i, x_j): support for a_i from x_j;
- S_{x_j,a_j}: array pointing to all values in other variables supported by (x_j, a_j);
- M: list of removed values.

Enforcing Arc Consistency: AC-4

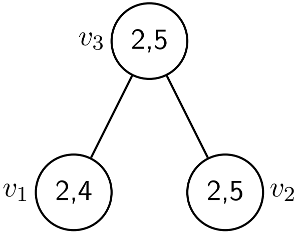
AC-4$(\mathcal{C})$:

Input: a constraint network $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, but arc-consistent network $\mathcal{C}'$

$M \leftarrow \emptyset$
initialize S_{x_i,a_i} and *counter*(x_i, a_i, x_j) for all R_{ij}
for each counter
    if *counter*$(x_i, a_i, x_j) = 0$
        then add (x_i, a_i) to *List*
    endif
endfor
while *List* is not empty
    choose and remove (x_i, a_i) from *List*, and add it to M
    for each (x_j, a_j) in S_{x_i,a_i}
        decrement *counter*(x_j, a_j, x_i)
        if *counter*$(x_j, a_j, x_i) = 0$
            then add (x_j, a_j) to *List*
        endif
    endfor
endwhile

Example: AC-4

Consider the same network as for AC-3.
Constraints: $v_3|v_1$ and $v_3|v_2$.



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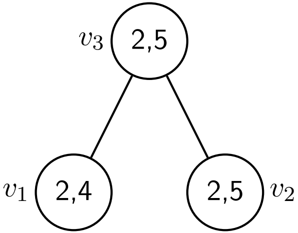

The initialization steps yield:

$$S_{v_3,2} = \{(v_1,2),(v_1,4),(v_2,2)\} \qquad S_{v_3,5} = \{(v_2,5)\}$$
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Example: AC-4

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$$S_{v_1,2} = \{(v_3,2)\} \qquad S_{v_1,4} = \{(v_3,2)\}$$

Furthermore:

$$\textit{counter}(v_3,2,v_1) = 2 \quad \text{and} \quad \textit{counter}(v_3,5,v_1) = 0.$$

All other counters are 1 (note: we only need consider counters between connected variables).

$$\textit{List} = \{(v_3,5)\} \quad \text{and} \quad M = \emptyset.$$

When $(v_3,5)$ is removed from *List* and added to M, we obtain *counter*$(v_2,5,v_3) = 0$ and add $(v_2,5)$ to *List*. Then $(v_2,5)$ is removed from *List* and added to M. $(v_2,5)$ is only supported by $(v_3,5)$, but that pair is already in M, and we are done.

Beyond Arc Consistency

- Sometimes "enforcing arc consistency" is sufficient for detecting inconsistent (unsolvable) networks; but ...
- enforcing arc consistency is not complete for deciding consistency of networks; because ...
- inferences rely only on domain constraints and single binary constraints defined on the domains.

$\Rightarrow$ We consider further concepts of local consistency

Path Consistency

Definition

(a) A binary constraint R_{ij} for variables v_i, v_j is path-consistent relative to a third variable v_k if for every pair $(a_i, a_j) \in R_{ij}$, there exists an $a_k \in D_k$ such that $(a_i, a_k) \in R_{ik}$ and $(a_k, a_j) \in R_{kj}$.

(b) A pair of distinct variables v_i, v_j is path-consistent relative to variable v_k if any instantiation a of $\{v_i, v_j\}$ with $(a(v_i), a(v_j)) \in R_{ij}$ can be extended to an instantiation a' of $\{v_i, v_j, v_k\}$ such that $(a'(v_i), a'(v_k)) \in R_{ik}$ and $(a'(v_k), a'(v_j)) \in R_{kj}$ ("extended" means: $a = a'|_{\{v_i, v_j\}}$).

(c) A set of distinct variables $\{v_i, v_j, v_k\}$ is path-consistent if any pair of these variables is path-consistent relative to the omitted third variable.

(d) A constraint network is path-consistent if all its three-element subsets of variables are path-consistent.

An Example

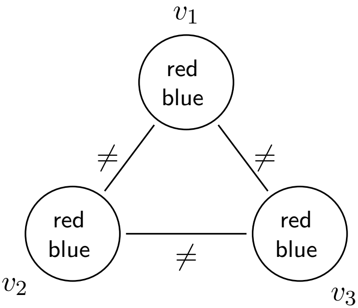

Figure: This network is arc-consistent, but not path-consistent.

Revising a Path

Revise-3$(\{v_i, v_j\}, v_k)$:

Input: a binary network $\langle V, D, C \rangle$ with variables v_i, v_j, v_k
Output: a revised constraint R_{ij} path-consistent with v_k

for each pair $(a_i, a_j) \in R_{ij}$
  if there is no $a_k \in D_k$ such that $(a_i, a_k) \in R_{ik}$
    and $(a_j, a_k) \in R_{jk}$
    then delete (a_i, a_j) from R_{ij}
  endif
endfor

This is equivalent to applying:

$$R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})$$

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$$R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})$$

Revising a Path: Properties

Lemma

When applied to a constraint network $\mathcal{C}$, procedure Revise-$3(\{v_i, v_j\}, v_k)$:

- *does not do anything if the pair v_i, v_j is path-consistent relative to v_k, and otherwise*
- *transforms the network into an equivalent form where the pair v_i, v_j is path-consistent relative to v_k.*

Proof.

From the definition of path consistency. □

Revising a Path: Complexity

Lemma

Let t be the maximal number of tuples in one of the binary constraints, and let k be an upper bound for the domain sizes.

The worst-case runtime of Revise-3 is $\mathcal{O}(t \cdot k)$.
The best-case runtime of Revise-3 is $\Omega(t)$.

Note that $t \leq k^2$, so the complexity of Revise-3 can also be expressed as $\mathcal{O}(k^3)$ in the worst and $\Omega(k^2)$ in the best case.

Enforcing Path Consistency: PC-1

PC-1$(\mathcal{C})$:

Input: a constraint network $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, path-consistent network $\mathcal{C}'$

repeat
  for each (ordered) triple of variables v_i, v_j, v_k:
    Revise-$3(\{v_i, v_j\}, v_k)$
  endfor
until no constraint is changed

Enforcing Path Consistency: Soundness of PC-1

Lemma

When applied to a constraint network $\mathcal{C}$, the PC-1 algorithm computes a path-consistent constraint network which is equivalent to $\mathcal{C}$.

Proof.

Follows directly from the properties of Revise-3. □

Enforcing Path Consistency: Complexity of PC-1

Lemma

Let $\mathcal{C}$ be a constraint network with n variables, each with a domain of size $\leq k$. Let t be an upper bound of the number of tuples in one of the binary constraints in $\mathcal{C}$.

The worst-case runtime of PC-1 on this network is $\mathcal{O}(n^5 \cdot t^2 \cdot k)$. The best-case runtime of PC-1 on this network is $\Omega(n^3 \cdot t)$.

Because $t \leq k^2$, the runtime bounds can also be stated as $\mathcal{O}(n^5 \cdot k^5)$ and $\Omega(n^3 \cdot k^2)$, respectively.

Enforcing Path Consistency: Complexity of PC-1

Proof (worst case).

In each iteration of the outer loop in PC-1, only one value pair might be deleted from one of the constraints. Hence the number of iterations may be as large as $\mathcal{O}(n^2 \cdot t)$.
Processing a specific triple of constraints (there are $\mathcal{O}(n^3)$ many such triples) costs $\mathcal{O}(t \cdot k)$.
Hence each iteration costs $\mathcal{O}(n^3 \cdot t \cdot k)$. □

Proof (best case).

In the best case, the network is already path-consistent and only one iteration through the outer loop is needed. There are $\Omega(n^3)$ calls to Revise-3, each requiring time $\Omega(t)$ in the best case. □

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Enforcing Path Consistency: PC-2

PC-2$(\mathcal{C})$:

Input: a constraint network $\mathcal{C} = \langle V, D, C \rangle$
Output: an equivalent, path-consistent network $\mathcal{C}'$

queue $\leftarrow \{(i,k,j) \,:\, 1 \leq i < j \leq n, 1 \leq k \leq n, k \neq i, k \neq j\}$
while *queue* is not empty
    select and delete a triple (i,k,j) from *queue*
    Revise-$3(\{v_i, v_j\}, v_k)$
    if R_{ij} has changed **then**
        queue $\leftarrow$ *queue* $\cup \{(l,i,j),\ (l,j,i) : 1 \leq l \leq n, l \neq i,j\}$
    endif
endwhile

Enforcing Path Consistency: Soundness of PC-2

Lemma

When applied to a constraint network $\mathcal{C}$, the PC-2 algorithm computes a path-consistent constraint network which is equivalent to $\mathcal{C}$.

Proof.

Equivalence follows directly from the properties of Revise-3. To see that the remaining constraint network is path-consistent, verify the following invariant:

> *Before and after each iteration of the* **while***-loop, for each pair v_i, v_j which is not path-consistent relative to v_k, one of the triples (i, k, j) and (j, k, i) is contained in the queue.*

□

Enforcing Path Consistency: Complexity of PC-2

Lemma

Let $\mathcal{C}$ be a constraint network with n variables, each with a domain of size $\leq k$. Let t be an upper bound of the number of tuples in one of the binary constraints in $\mathcal{C}$.

The worst-case runtime of PC-2 on this network is $\mathcal{O}(n^3 \cdot t^2 \cdot k)$. The best-case runtime of PC-2 on this network is $\Omega(n^3 \cdot t)$.

Because $t \leq k^2$, the runtime bounds can also be stated as $\mathcal{O}(n^3 \cdot k^5)$ and $\Omega(n^3 \cdot k^2)$, respectively.

Enforcing Path Consistency: Complexity of PC-2

Proof (worst case).

There are initially $\mathcal{O}(n^3)$ elements in the queue. Whenever some constraint R_{ij} is reduced, which can happen at most $\mathcal{O}(n^2 \cdot t)$ many times, $O(n)$ elements are added to the queue. Thus, the total number of elements added to the queue is bounded by $\mathcal{O}(n^3 \cdot t)$.
Each iteration of the **while** loop removes an element from the queue, so there are at most $\mathcal{O}(n^3 \cdot t)$ iterations and hence at most $\mathcal{O}(n^3 \cdot t)$ calls to Revise-3, each requiring time $\mathcal{O}(t \cdot k)$, for a total runtime bound of $\mathcal{O}(n^3 \cdot t^2 \cdot k)$. □

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Similar to PC-1. □

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Similar to PC-1. □

Arc and Path Consistency: Overview

	Worst Case	Best Case
AC-1	$\mathcal{O}(n \cdot k \cdot e \cdot t)$	$\Omega(e \cdot k)$
AC-3	$\mathcal{O}(e \cdot k \cdot t)$	$\Omega(e \cdot k)$
AC-4	$\mathcal{O}(e \cdot t)$	$\Omega(e \cdot k^2)$
PC-1	$\mathcal{O}(n^5 \cdot t^2 \cdot k)$	$\Omega(n^3 \cdot t)$
PC-2	$\mathcal{O}(n^3 \cdot t^2 \cdot k)$	$\Omega(n^3 \cdot t)$
PC-4*	$\mathcal{O}(n^3 \cdot t \cdot k)$	$\Omega(n^3 \cdot t \cdot k)$

*not discussed in this lecture

Remark: $\mathcal{O}(n^3 \cdot t \cdot k)$ is the optimal (worst-case) runtime for enforcing path consistency, i.e., there are (arbitrarily large) constraint networks for which no better algorithm exists.

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Higher Levels of i-Consistency

The local consistency notions presented so far can be roughly summarized as follows:

- Arc consistency: Every consistent assignment to a single variable can be consistently extended to any second variable.
- Path consistency: Every consistent assignment to two variables can be consistently extended to any third variable.

(Side remark: This is a bit of an oversimplification because we ignored k-ary constraints with $k \geq 3$ so far. More on this later.)

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i-Consistency

Let $\mathcal{C} = \langle V, D, C \rangle$ be a constraint network.

Definition

(a) A relation $R_S \in C$ with scope S of size $i-1$ is *i-consistent* relative to variable $v_i \notin S$ if for every tuple $t \in R_S$, there exists an $a \in D_i$ such that (t, a) is consistent.

(b) A constraint network is *i-consistent* if any consistent instantiation of $i-1$ (distinct) variables $v_1, \ldots, v_{i-1}$ of the network can be extended to a *consistent* instantiation of the variables $v_1, \ldots, v_i$, where v_i is any variable in V distinct from $v_1, \ldots, v_{i-1}$.

Global Consistency

Definition

- A network $\mathcal{C}$ is strongly i-consistent if it is j-consistent for each $j \leq i$.
- A network $\mathcal{C}$ with n variables is globally consistent if it is strongly n-consistent.

Note: Solutions to globally consistent networks can be found without search. (How?)

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Arc/Path Consistency vs. 2/3-Consistency

Note:

- 2-consistency coincides with arc consistency.
- For networks containing binary constraints only, 3-consistency coincides with path consistency.
- Each 3-consistent network is path-consistent.
- The converse is not true: For networks with constraints of arity ≥ 3, 3-consistency is stricter than path consistency.

3-Consistency: Examples

Example

$V = \{v_1, v_2, v_3\}$
$D_1 = D_2 = D_3 = \{0, 1\}$
$R_{123} = \{(0, 0, 0)\}$

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$R_{12} = R_{13} = R_{23} = \{(0, 1), (1, 0), (1, 1)\}$

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Revise-i

Revise-$i(\{v_1, \ldots, v_{i-1}\}, v_i)$:

Input: a network $\langle V, D, C\rangle$ and a constraint R_S with scope $S = \{v_1, \ldots, v_{i-1}\}$
Output: a constraint R_S which is i-consistent rel. to v_i

for each instantiation $\overline{a}_{i-1} \in R_S$
    if there is no $a_i \in D_i$ such that $(\overline{a}_{i-1}, a_i)$
        is consistent
        then delete $\overline{a}_{i-1}$ from R_S
    endif
endfor

- R_S can be the universal relation wrt. S.
- If the input network is binary, then Revise-i runs in time $\mathcal{O}(k^i)$.
- In general, Revise-i runs in time $\mathcal{O}((2 \cdot k)^i)$, since $\mathcal{O}(2^i)$ constraints must be processed for each tuple.

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i-Consistency: Algorithm

Enforce i-Consistency$(\mathcal{C})$:

Input: A constraint network $\mathcal{C} = \langle V, D, C \rangle$.
Output: An i-consistent network equivalent to $\mathcal{C}$.

repeat
  for each subset of $S \subseteq V$ of size $i - 1$ and each $v_i \notin S$
    Revise-$i(\{v_1, \ldots, v_{i-1}\}, v_i)$
  endfor
until no constraint is changed

The Revise-i call can equivalently be stated as follows:
Let $\mathcal{S}$ be the set of all subsets of $\{v_1, \ldots, v_i\}$ that contain v_i and occur as scopes of some constraint in the network. Then apply

$$R_S \leftarrow R_S \cap \pi_S(\bowtie_{S' \in \mathcal{S}} R_{S'}).$$

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i-Consistency: Complexity

Lemma

Let $\mathcal{C}$ be a constraint network with n variables, each with a domain of size $\leq k$. When applied to $\mathcal{C}$, the "Enforce i-Consistency" algorithm runs in time $\mathcal{O}(2^i \cdot (n \cdot k)^{2i-1})$.

Proof.

Each call to Revise-i requires time $\mathcal{O}((2 \cdot k)^i)$. In each iteration of the outer loop, $\mathcal{O}(n^i)$ combinations of S and v_i need to be processed. If only one tuple is removed from one constraint in each iteration up to the final one, the outer loop may need to iterate $\mathcal{O}(n^{i-1} \cdot k^{i-1})$ times.
This leads to an overall runtime of $\mathcal{O}(2^i \cdot (n \cdot k)^{2i-1})$. □

Note: Improvements similar to AC-4 and PC-4 exist and achieve a worst-case runtime of $\mathcal{O}(n^i \cdot k^i)$.

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i-Consistency: Comparison to AC-x and PC-x

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improved i-consistency*, $i = 2$	$\mathcal{O}(n^2 \cdot k^2)$
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PC-1	$\mathcal{O}(n^5 \cdot t^2 \cdot k) = \mathcal{O}(n^5 \cdot k^5)$
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Remark: $\mathcal{O}(n^i \cdot k^i)$ is the optimal (worst-case) runtime for enforcing i-consistency, i.e., there are (arbitrarily large) constraint networks for which no better algorithm exists.

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Extensions of Arc Consistency

- General i-consistency is powerful, but expensive to enforce.
- Usually, arc consistency and path consistency offer a good compromise between pruning power and computational overhead.
- However, they are of limited usefulness for constraints on more than two variables.

Example

Consider a constraint network with three integer variables $v_1, v_2, v_3 \geq 0$ and the constraints $v_3 \geq 13$ and $v_1 + v_2 + v_3 \leq 15$.
We should be able to infer $v_1 \leq 2$ and $v_2 \leq 2$, but regular arc consistency is not enough!

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$\rightsquigarrow$ Consider generalizations of arc consistency to non-binary constraints.

Generalized Arc Consistency

Let $\mathcal{C} = \langle V, D, C \rangle$ be a constraint network.

Definition

(a) A variable v_i is (generalized) arc-consistent relative to a constraint $R \in C$ whose scope contains v_i if for every value $a_i \in D_i$ there exists a tuple $\overline{a} \in R$ with $\overline{a}_i = a_i$.

(b) A constraint $R \in C$ is (generalized) arc-consistent iff all variables in its scope are generalized arc-consistent relative to R.

(c) A network $\mathcal{C}$ is (generalized) arc-consistent if all its constraints are generalized arc-consistent.

Generalized Arc Consistency: Update Rule

To enforce generalized arc consistency, repeatedly apply

$$D_i \leftarrow D_i \cap \pi_i(R_S \bowtie D_{S \setminus \{v_i\}})$$

Note how this generalizes the usual arc consistency update rule:

$$D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$$

Alternatives to Generalized Arc Consistency

- Like arc consistency, generalized arc consistency propagates constraints by considering a single constraint at a time.
- In particular, it considers how assignments to each individual variable are restricted by the values allowed for the other variables participating in the constraint.
- Alternatively, we can consider how each individual variable restricts the values allowed for the other variables participating in the constraint:

$$R_{S \setminus \{v_i\}} \leftarrow R_{S \setminus \{v_i\}} \cap \pi_{S \setminus \{v_i\}}(R_S \bowtie D_i)$$

  (relational arc consistency)
- Note that in the case of binary constraints, these two cases are the same, so both approaches are natural generalizations of (binary) arc consistency.

Generalizations of Arc Consistency: Comparison

$$\text{AC: } D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$$

$$\text{generalized AC: } D_i \leftarrow D_i \cap \pi_i(R_S \bowtie D_{S \setminus \{v_i\}})$$

$$\text{relational AC: } R_{S \setminus \{v_i\}} \leftarrow R_{S \setminus \{v_i\}} \cap \pi_{S \setminus \{v_i\}}(R_S \bowtie D_i)$$

Example

Consider a constraint network with three integer variables $v_1, v_2, v_3 \geq 0$ and the constraints $v_3 \geq 13$ and $v_1 + v_2 + v_3 \leq 15$.

- Generalized AC infers $v_1 \leq 2$, $v_2 \leq 2$.
- Relational AC infers $v_1 + v_2 \leq 2$.

Generalizations of Arc Consistency: Comparison

$$\text{AC: } D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$$

$$\text{generalized AC: } D_i \leftarrow D_i \cap \pi_i(R_S \bowtie D_{S\setminus\{v_i\}})$$

$$\text{relational AC: } R_{S\setminus\{v_i\}} \leftarrow R_{S\setminus\{v_i\}} \cap \pi_{S\setminus\{v_i\}}(R_S \bowtie D_i)$$

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Generalizations of Arc Consistency: Comparison

$$\text{AC: } D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j)$$
$$\text{generalized AC: } D_i \leftarrow D_i \cap \pi_i(R_S \bowtie D_{S\setminus\{v_i\}})$$
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Generalizations of Arc Consistency: Comparison

$$
\begin{aligned}
\text{AC: } & D_i \leftarrow D_i \cap \pi_i(R_{ij} \bowtie D_j) \\
\text{generalized AC: } & D_i \leftarrow D_i \cap \pi_i(R_S \bowtie D_{S \setminus \{v_i\}}) \\
\text{relational AC: } & R_{S \setminus \{v_i\}} \leftarrow R_{S \setminus \{v_i\}} \cap \pi_{S \setminus \{v_i\}}(R_S \bowtie D_i)
\end{aligned}
$$

Example

Consider a constraint network with three integer variables $v_1, v_2, v_3 \geq 0$ and the constraints $v_3 \geq 13$ and $v_1 + v_2 + v_3 \leq 15$.

- Generalized AC infers $v_1 \leq 2$, $v_2 \leq 2$.
- Relational AC infers $v_1 + v_2 \leq 2$.

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