Constraint Satisfaction Problems

Constraint Networks

Bernhard Nebel and **Stefan Wölfl**

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Malte Helmert and Stefan Wölfl
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Constraint Networks

Definition

A constraint network is a triple

$$\mathcal{C} = \langle V, \mathrm{dom}, C \rangle$$

where:

- V is a non-empty and finite set of variables.
- dom is a function that assigns a non-empty (value) set (domain) to each variable $v \in V$.
- C is a set of relations over variables of V (constraints), i.e., each constraint is a relation $R_{v_1,\dots,v_n}$ over some variables $v_1, \dots, v_n$ in V.

The set of scopes $\{S_1, \dots S_t\}$ is called network scheme.

Example: 4-Queens Problem

Consider variables $v_1, \dots, v_4$ (associated to the columns of a 4×4-chess board). Each of these variables v_i has as its domain $\{1, \dots, 4\}$ (conceived of as the row positions of a queen in column i).

	v_1	v_2	v_3	v_4
1				
2				
3				
4				

Define then binary constraints (thus encoding possible queen movements):

$$R_{v_1,v_2} := \{(1,3),(1,4),(2,4),(3,1),(4,1),(4,2)\}$$

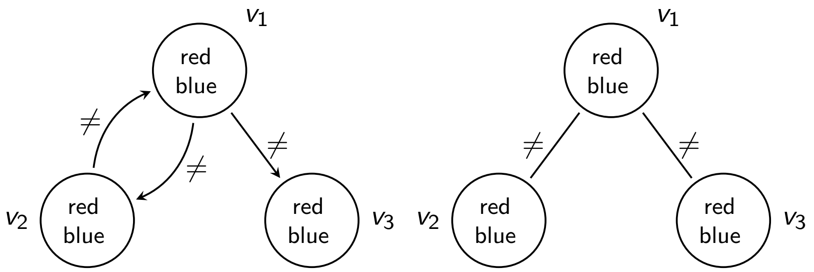
$$R_{v_1,v_3} := \{(1,2),(1,4),(2,1),(2,3),(3,2),(3,4),(4,1),(4,3)\}$$

...

Graph Representation of Binary Constraint Networks

Constraint networks with binary constraints only can be represented by a directed labelled graph
(even: an undirected graph if all constraints are symmetric).

Example: The constraint network defined by:
$V = \{v_1, v_2, v_3\}$,
$\mathrm{dom}(v_i) = \{\text{red}, \text{blue}\}$ $(i = 1..3)$,
$C = \{((v_1, v_2), \neq), ((v_2, v_1), \neq), ((v_1, v_3), \neq)\}$



Solvability of Networks

Definition

A constraint network is solvable (or: satisfiable) if there exists an assignment

$$a\colon V \to \bigcup_{v \in V} \mathrm{dom}(v)$$

such that

(a) $a(v) \in \mathrm{dom}(v)$, for each $v \in V$,

(b) $(a(v_1), \ldots, a(v_n)) \in R_{v_1,\ldots,v_n}$ for all constraints $R_{v_1,\ldots,v_n}$.

A solution of a constraint network is an assignment that solves the network.

Instantiation, Partial Solution

Let $\mathcal{C} = \langle V, \mathrm{dom}, C\rangle$ be a constraint network.

Definition

(a) An instantiation of a subset V' of V is an assignment $a : V' \to \bigcup_{v \in V'} \mathrm{dom}(v_i)$ with $a(v_i) \in \mathrm{dom}(v_i)$.

(b) An instantiation a is a partial solution if a satisfies each constraint with scope $S \subseteq V'$.
We also say: a is consistent relative to $\mathcal{C}$.

(c) For an instantiation a of a subset $V' = \{v_1, \ldots, v_n\}$ and a constraint R with scope $S \subseteq V'$, let

$$\overline{a}[S] := (a(v_1), \ldots, a(v_n)).$$

Hence a solution is an instantiation of all variables in V that is consistent relative to $\mathcal{C}$.

Instantiation, Solution

Note:

(a) An instantiation of variables in $V' \subseteq V$, a, is a partial solution (consistent relative to $\mathcal{C}$) iff

$$\overline{a}[S] \in R, \quad \text{for each constraint } R \text{ with scope } S \subseteq V'.$$

(b) Not every partial solution is part of a (full) solution, i.e., there may be partial solutions of a constraint network that cannot be extended to a solution. For the 4-queens problem, for example,

	v_1	v_2	v_3	v_4
1	q			
2			q	
3				
4		q		

Normalized Constraint Networks

Let $\mathcal{C} = \langle V, \mathrm{dom}, C \rangle$ be a constraint network.
According to our definition it is possible that C contains constraints

$$R_{v_{i_1},\ldots,v_{i_k}} \quad \text{and} \quad S_{v_{j_1},\ldots,v_{j_k}}$$

where $(j_1,\ldots,j_k)$ is just a permutation of $(i_1,\ldots,i_k)$.
Without changing the set of solutions, we can simplify the network by deleting $S_{v_{j_1},\ldots,v_{j_k}}$ from C and rewriting $R_{v_{i_1},\ldots,v_{i_k}}$ as follows:

$$R_{v_{i_1},\ldots,v_{i_k}} \leftarrow R_{v_{i_1},\ldots,v_{i_k}} \cap \pi_{v_{i_1},\ldots,v_{i_k}}(S_{v_{j_1},\ldots,v_{j_k}}).$$

Given a fixed order on the set of variables V, we can systematically delete-and-refine constraints. The result is a constraint network that contains *at most one constraint for each subset of variables*. This network is referred to as a normalized constraint network.

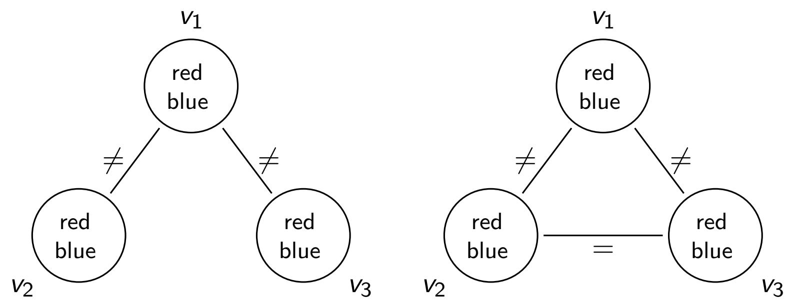
Equivalence

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks on the same set of variables and on the same domains for each variable.

Definition

$\mathcal{C}$ and $\mathcal{C}'$ are equivalent if each solution of $\mathcal{C}$ is a solution of $\mathcal{C}'$, and vice versa.

Example:



Tightness

Let $\mathcal{C}$ and $\mathcal{C}'$ be (normalized) constraint networks on the same set of variables and on the same domains for each variable.

Definition

$\mathcal{C}$ is as tight as $\mathcal{C}'$ if for each constraint R of $\mathcal{C}$ with scope S,

(a) $\mathcal{C}'$ has no constraint with scope S, or

(b) $R \subseteq R'$, where R' is the constraint of $\mathcal{C}'$ with scope S.

Notes:

- Constraint tightness has a large influence on the solubility of constraint networks.
- Be warned: different concepts of tightness can be found in the literature
- Here: Tightness does not account for comparing constraints with different arities

Intersection of Networks

Let $\mathcal{C}$ and $\mathcal{C}'$ be constraint networks as above.

Definition

The intersection of $\mathcal{C}$ and $\mathcal{C}'$, $\mathcal{C} \cap \mathcal{C}'$, is the network defined by intersecting for each scope S of constraints $R_S \in C$ and $R'_S \in C'$ the respective relations, i.e.,

$$R''_S := R_S \cap R'_S.$$

If for a scope S only one of the networks contains a constraint, then we set:

$$R''_S := R_S \quad (\text{or } := R'_S, \text{ resp.})$$

Lemma

If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent networks, then $\mathcal{C} \cap \mathcal{C}'$ is equivalent to both networks and as tight as both networks.

Minimal Network

Definition

Let $\mathcal{C}_0$ be a constraint network and let $\mathcal{C}_1 \ldots, \mathcal{C}_k$ be the set of *all* constraint networks (defined on the same set of variables and the same domains) that are equivalent to $\mathcal{C}_0$.

$$\bigcap_{1 \leq i \leq k} \mathcal{C}_i$$

is the minimal network of $\mathcal{C}_0$.

Lemma

The minimal network is equivalent to and as tight as all the constraint networks $\mathcal{C}_i$. There is no network equivalent to $\mathcal{C}_0$ that is tighter than the minimal network.

Projecting Relations

Let R_S be a relation with scope $S = \{v_1, \ldots, v_k\}$ (we can think of R_S as a constraint network ...).

Definition

The projection network of R_S, $\mathrm{Proj}(R_S)$, is the constraint network defined by:

$$\begin{aligned} V &:= S \\ \mathrm{dom}(v_i) &:= \pi_{v_i}(R_S) \\ R_{v_i,v_j} &:= \pi_{v_i,v_j}(R_S) \end{aligned}$$

Note: The projection network is an upper approximation by binary networks in the following sense:

Lemma

Any solution of R_S (as a network) defines a solution of $\mathrm{Proj}(R_S)$.

Binary Representation

Definition

A relation R_S with scope S has a binary representation if the relation (conceived of as a network) is equivalent to $\mathrm{Proj}(R_S)$.

From the fact that a relation has a binary representation, it does not follow that all its projections have binary representations as well (Exercise!).

Definition

A relation R_S with scope S is binary decomposable if the relation itself and all its projections to subsets of S (with at least 3 elements) have a binary representation.

Primal Constraint Graphs

Let $\mathcal{C} = \langle V, \mathrm{dom}, C\rangle$ be a (normalized) constraint network.

Definition

The primal constraint graph of a network $\mathcal{C} = \langle V, \mathrm{dom}, C\rangle$ is the undirected graph

$$G_{\mathcal{C}} := \langle V, E_{\mathcal{C}}\rangle$$

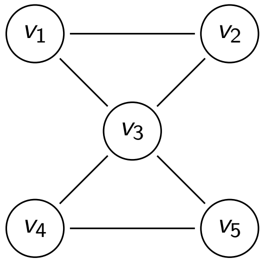
where

$$\{u, v\} \in E_{\mathcal{C}} \iff \{u, v\} \text{ is a subset of the scope of some constraint in } \mathcal{C}.$$

Primal Constraint Graph: Example

Consider a constraint network with variables $v_1, \dots, v_5$ and two ternary constraints R_{v_1,v_2,v_3} and S_{v_3,v_4,v_5}.

Then the primal constraint graph of the network has the form:



Absence of an edge between two variables/nodes means that there is no direct constraint between these variables.

Dual Constraint Graphs

Definition

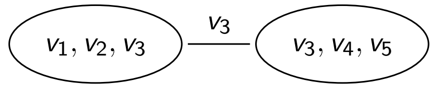
The dual constraint graph of a constraint network $\mathcal{C} = \langle V, \mathrm{dom}, C \rangle$ is the labeled graph

$$D_{\mathcal{C}} := \langle V', E_{\mathcal{C}}, l \rangle$$

with

$$X \in V' \iff X \text{ is the scope of some constraint in } \mathcal{C}$$
$$\{X, Y\} \in E_{\mathcal{C}} \iff X \cap Y \neq \emptyset$$
$$l : E_{\mathcal{C}} \to 2^V, \quad \{X, Y\} \mapsto X \cap Y$$

In the example above, the dual constraint graph is:



Constraint Hypergraph

Definition

The constraint hypergraph of a constraint network $\mathcal{C} = \langle V, \mathrm{dom}, C \rangle$ is the hypergraph

$$H_{\mathcal{C}} := \langle V, E_{\mathcal{C}} \rangle$$

with

$$X \in E_{\mathcal{C}} \iff X \text{ is the scope of some constraint in } \mathcal{C}.$$

In the example above (constraint network with variables $v_1, \dots, v_5$ and two ternary constraints R_{v_1,v_2,v_3} and S_{v_3,v_4,v_5}) the hyperedges of the constraint hypergraph are:

$$E_{\mathcal{C}} = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}\}.$$

Simple Solution Strategy: Guess and Check

Backtracking: search systematically for consistent partial instantiations in a depth-first manner:

- forward phase: extend the current partial solution by assigning a consistent value to some new variable (if possible)
- backward phase: if no consistent instantiation for the current variable exists, we return to the previous variable.

Backtracking Algorithm

Backtracking$(\mathcal{C}, a)$:

Input: a constraint network $\mathcal{C} = \langle V, D, C \rangle$ and
a partial assignment a of $\mathcal{C}$
(e.g., the empty instantiation $a = \{\,\}$)
Output: a solution of $\mathcal{C}$ or "inconsistent"

if a is not consistent with $\mathcal{C}$:
  return "inconsistent"
if a is defined for all variables in V:
  return a
select some variable v_i for which a is not defined
for each value x from D_i:
  $a' := a \cup \{v_i \mapsto x\}$
  $a'' \leftarrow \text{Backtracking}(\mathcal{C}, a')$
  if a'' is not "inconsistent":
    return a''
return "inconsistent"

Literature

Rina Dechter.
Constraint Processing,
Chapter 2, Morgan Kaufmann, 2003