Constraint Satisfaction Problems

Mathematical Background: Sets, Relations, and Graphs

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Constraints, Sets, Relations, Graphs

- Formal definition of CSP uses sets and constraints
- Constraints are specific relations that restrict possible solutions
- CSP solving techniques use operations that manipulate sets and relations
- CSP instances can also be represented by various kinds of graphs
- Graph-theoretical notions can be used to describe, e.g., structural properties of constraint networks
- Complexity for solving CSP instances can depend on both the relations used in the constraints and properties of the constraint graphs

Sets

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Naive understanding:

a set is a "well-defined" collection of objects.

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Principles/Set-theoretical axioms (ZF):

- **Extensionality:** Two sets are equal if and only if they contain the same elements.
- **Empty set:** There is a set, $\emptyset$, with no elements.
- **Pairs:** For any pair of sets x, y, $\{x, y\}$ is a set.
- **Union:** For any set x, there exists a set, $\bigcup x$, whose elements are precisely the elements of the elements of x.
- ...

Sets

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Naive understanding:

a set is a "well-defined" collection of objects.

Principles/Set-theoretical axioms (ZF):

- ...
- **Separation:** For any set x and any property $F(y)$, there is a subset of x, $\{y \in x : F(y)\}$, containing precisely the elements y of x for which $F(y)$ holds.
- **Power set:** For any set x there exists a set 2^x such that the elements of 2^x are precisely the subsets of x.
- ... (axiom of foundation, axiom of replacement, infinite set axiom, axiom of choice)

Set-theoretical Notations:

Boolean operations on sets:

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B := \{x \in A : x \in B\}$$
$$A \setminus B := \{x \in A : x \notin B\}$$

Power set: $A \subseteq B$, $A \subsetneq B$, etc., are defined as usual.

$$2^A := \{B : B \subseteq A\}$$

(Ordered) pairs:

$$(x, y) := \{\{x\}, \{x, y\}\}$$
$$(x_1, \ldots, x_n) := ((x_1, \ldots, x_{n-1}), x_n)$$
$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$$

Boolean Algebra

Definition

A Boolean algebra (with complements) is a set A with

- two binary operations $\sqcap$, $\sqcup$,
- a unary operation $-$, and
- two distinct elements 0 and 1

such that for all elements a, b and c of A:

$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$	$a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$	Ass
$a \sqcup b = b \sqcup a$	$a \sqcap b = b \sqcap a$	Com
$a \sqcup (a \sqcap b) = a$	$a \sqcap (a \sqcup b) = a$	Abs
$a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$	$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$	Dis
$a \sqcup -a = 1$	$a \sqcap -a = 0$	Compl

Sets and Boolean Algebras

Definition

A set algebra on a set X is a non-empty subset of 2^X that is closed under unions, intersections, and complements.

Note: a set algebra on X contains X and $\emptyset$ as elements.

Lemma

(a) *The power set of a set is a set algebra.*

(b) *Each set algebra defines a Boolean algebra.*

Relations

Definition

A relation over sets $X_1, \ldots, X_n$ is a subset

$$R \subseteq X_1 \times \cdots \times X_n =: \prod_{1 \le i \le n} X_i.$$

The number n is referred to as arity of R.
An n-ary relation on a set X is a subset

$$R \subseteq X^n := X \times \cdots \times X \quad (n \text{ times}).$$

Since relations are sets, set-theoretical operations (union, intersection, complement) can be applied to relations as well.

Binary Relations

For binary relations on a set X we have some special operations:

Definition

Let R, S be binary (2-ary) relations on X.
The converse of relation R is defined by:

$$R^{-1} := \{(x, y) \in X^2 : (y, x) \in R\} .$$

The composition of relations R and S is defined by:

$$R \circ S := \{(x, z) \in X^2 : \exists y \in X \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S\} .$$

The identity relation is:

$$\Delta_X := \{(x, y) \in X^2 : x = y\} .$$

Operating on Binary Relations

Lemma

Let X be a non-empty set. Let $\mathcal{R}(X)$ be the set of all binary relations on X. Then:

(a) *$\mathcal{R}(X)$ is a set algebra on $X \times X$.*

(b) *For all relations $R, S, T \in \mathcal{R}(X)$:*

$$R \circ (S \circ T) = (R \circ S) \circ T$$

$$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

$$\Delta_X \circ R = R \circ \Delta_X = R$$

$$(R^{-1})^{-1} = R \text{ and } (-R)^{-1} = -(R^{-1})$$

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

$$(R \circ S) \cap T^{-1} = \emptyset \text{ if and only if } (S \circ T) \cap R^{-1} = \emptyset$$

Constraints: Relations over Variables

Let V be a set of variables. For $v \in V$, let $\mathrm{dom}(v)$ be a non-empty set (of values) (the domain of v).

Definition

A relation over (pairwise distinct) variables $v_1, \ldots, v_n \in V$ is an $n+1$-tuple

$$R_{v_1,\ldots,v_n} := (v_1, \ldots, v_n, R)$$

where R is a relation over $\mathrm{dom}(v_1), \ldots, \mathrm{dom}(v_n)$.

The sequence $(v_1, \ldots, v_n)$ is referred to as the range, the set $\{v_1, \ldots, v_n\}$ as the scope, and R as the graph of $R_{v_1,\ldots,v_n}$.

We will not always distinguish between the relation and its graph, e. g., we write

$$R_{v_1,\ldots,v_n} \subseteq \mathrm{dom}(v_1) \times \cdots \times \mathrm{dom}(v_n).$$

Selections, ...

Let $\overline{v} := (v_1, \ldots, v_n)$ and $R_{\overline{v}}$ be a relation over $\overline{v}$.

Definition

For fixed values $a_1 \in \mathrm{dom}(v_{i_1}), \ldots, a_k \in \mathrm{dom}(v_{i_k})$,

$$\sigma_{v_{i_1}=a_1,\ldots,v_{i_k}=a_k}(R_{\overline{v}}) := \{(x_1, \ldots, x_n) \in R_{\overline{v}} : x_{i_j} = a_j, 1 \leq j \leq k\}$$

defines a relation over $\overline{v}$.

The (unary) operation $\sigma_{v_{i_1}=a_1,\ldots,v_{i_k}=a_k}$ is called selection or restriction.

. . . Projections, . . .

Let $\overline{v} := (v_1, \ldots, v_n)$ be as above, and let $(i_1, \ldots, i_k)$ be a k-tuple of pairwise distinct elements of $\{1, \ldots, n\}$ $(k \leq n)$. For $\overline{x} = (x_1, \ldots, x_n)$, set $\overline{x}_{i_1,\ldots,i_k} := (x_{i_1}, \ldots, x_{i_k})$.

Definition

For a relation $R_{\overline{v}}$ over $\overline{v}$,

$$\pi_{v_{i_1},\ldots,v_{i_k}}(R_{\overline{v}}) := \left\{\overline{y} \in \prod_{1 \leq j \leq k} \mathrm{dom}(v_{i_j}) \;:\; \overline{y} = \overline{x}_{i_1,\ldots,i_k}, \text{ for some } \overline{x} \in R_{\overline{v}}\right\}$$

is a relation over $\overline{v}_{i_1,\ldots,i_k}$, the projection of $R_{\overline{v}}$ on $\overline{v}_{i_1,\ldots,i_k}$.

Note: For binary relations $R = R_{x,y}$, $R^{-1} = \pi_{y,x}(R_{x,y})$.

. . . Joins

For tuples $\overline{x}$ and $\overline{y}$ define:

- $\overline{x} - \overline{y}$: the subsequence of elements in $\overline{x}$ that do not occur in $\overline{y}$.
- $\overline{x} \cap \overline{y}$: the subsequence of $\overline{x}$ with elements that occur in $\overline{y}$.
- $\overline{x} \cup \overline{y}$: the sequence resulting from $\overline{x}$ by adding $\overline{y} - \overline{x}$.

Definition

Let $R_{\overline{v}}$ and $S_{\overline{w}}$ be relations over variables $\overline{v}$ and $\overline{w}$, resp.

$$R_{\overline{v}} \bowtie S_{\overline{w}} := \left\{ \overline{x} \cup \overline{y} \ : \ \overline{x} \in R_{\overline{v}},\ \overline{y} \in R_{\overline{w}},\ \text{and } \overline{x}_{\overline{v} \cap \overline{w}} = \overline{y}_{\overline{v} \cap \overline{w}} \right\}$$

is a relation over $\overline{v} \cup \overline{w}$, the join of $R_{\overline{v}}$ and $S_{\overline{w}}$.

Note: For binary relations $R = R_{x,y}$ and $S = S_{y,z}$ on the same set,

$$R \circ S = \pi_{x,z}(R_{x,y} \bowtie S_{y,z}).$$

Examples

Consider relations $R := R_{x_1,x_2,x_3}$ and $S := S_{x_2,x_3,x_4}$ defined by:

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

Then $\sigma_{x_3=c}(R)$, $\pi_{x_2,x_3}(R)$, $\pi_{x_2,x_1}(R)$, and $R \bowtie S$ are:

x_1	x_2	x_3
b	b	c
c	b	c

x_2	x_3
b	c
b	c
n	n

x_2	x_1
b	b
b	c
n	c

x_1	x_2	x_3	x_4
b	b	c	2
b	b	c	3
c	b	c	2
c	b	c	3

Undirected Graph

Definition

An (undirected, simple) graph is an ordered pair

$$G := \langle V, E \rangle$$

where:

- V is a finite set (of vertices, nodes);
- E is a set of two-element subsets of (not necessarily distinct) nodes (called edges).

The order of a graph is the number of vertices $|V|$.
The size of a graph is the number of edges $|E|$.
The degree of a vertex is the number of vertices to which it is connected by an edge.

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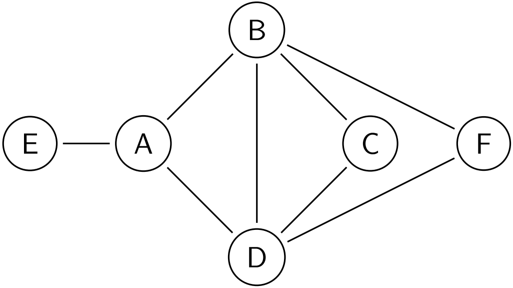
Graph: Example



Graph: Definitions

Definition

Let $G = \langle V, E \rangle$ be an undirected graph.

(a) If $e = \{u, v\} \in E$, then u and v are called adjacent (or: connected by e).

(b) A path in G is a sequence of vertices $v_0, \ldots, v_k$ such that $\{v_{i-1}, v_i\} \in E$ $(1 \leq i \leq k)$. k is the length, v_0 is the start vertex, and v_k is the end vertex of the path.

(c) A cycle is a path $v_0, \ldots, v_k$ with $v_0 = v_k$.

(d) A path $v_0, \ldots, v_k$ is simple if $v_i \neq v_j$ for all $i \neq j$.

(e) A cycle $v_0, \ldots, v_k$ is simple if $v_i \neq v_j$ for all $i, j \geq 1, i \neq j$.

Graph: Definitions

Let $G = \langle V, E \rangle$ be an undirected graph.

Definition

(a) G is connected if for each pair of vertices u and v, there exists a path from u to v.

(b) G is a tree if G is cycle-free.

(c) G is complete if any pair of vertices is connected by an edge.

Definition

Let S be a subset of V. Then $G_S := \langle S, E_S \rangle$ is called the subgraph relative to S, where $E_S := \{\{u, v\} \in E : u, v \in S\}$.

Definition

A clique in a graph G is a complete subgraph of G.

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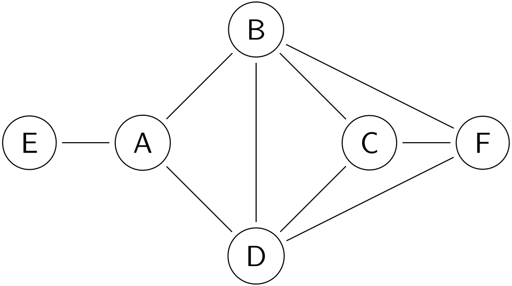

Figure: Example

Examples

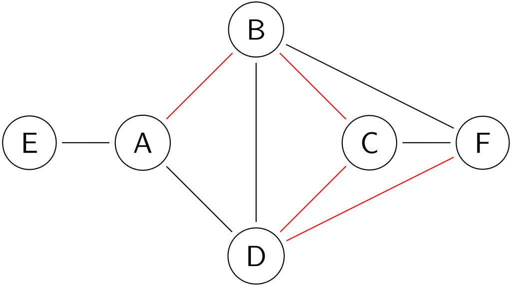

Figure: A path A,B,C,D,F

Examples

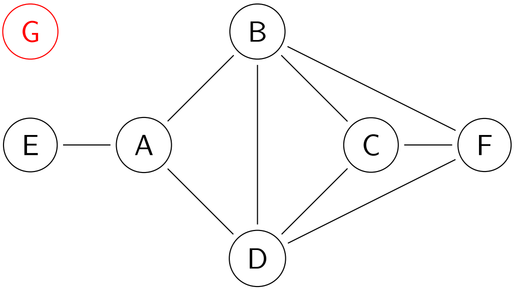

Figure: A non-connected and incomplete graph

Examples

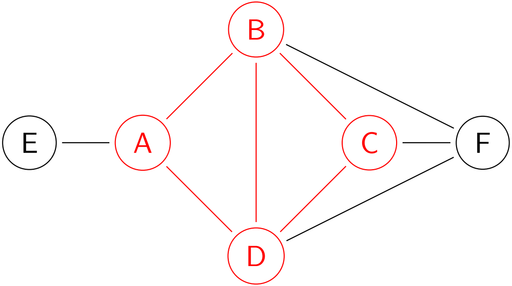

Figure: A subgraph

Examples

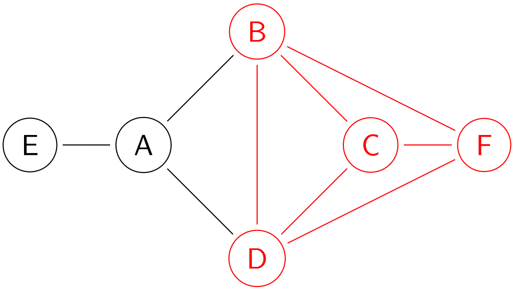

Figure: A clique

Directed Graph

Definition

A (simple) directed graph (or: digraph) is an ordered pair

$$G := \langle V, A \rangle$$

where:

- V is a set (of vertices or nodes),
- A is a set of (ordered) pairs of vertices (or: arcs, edges, or arrows).

The number of edges with a vertex v as start vertex is called the outdegree of v; the number of edges with v as end vertex is the indegree of v.

Nodes that point to v are called parents, nodes to which an edge from v points are called child nodes.

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Nodes that point to v are called parents, nodes to which an edge from v points are called child nodes.

Digraph: Definitions

Definition

Let $G = \langle V, A \rangle$ be a directed graph.

(a) A (directed) path is a sequence of arcs $e_1, \ldots, e_k$ such that the end vertex of e_i is the start vertex of e_{i+1} (analogously, (directed) cycle).

(b) A digraph is strongly connected if each pair of nodes u, v is connected by a directed path from u to v.

(c) A digraph is acyclic if it has no directed cycles.

Digraph: Example

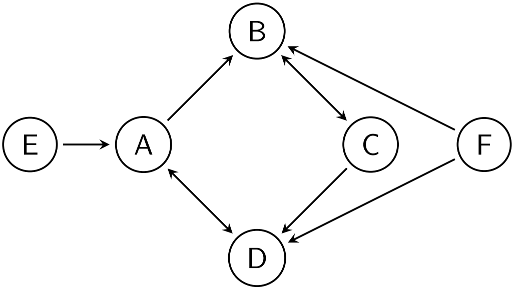

Figure: A directed graph with a strongly connected subgraph

Hypergraph

Graphs can be used to represent binary relations between nodes. For relations of higher arity we need:

Definition

A hypergraph is a pair

$$H := \langle V, E \rangle$$

where

- V is a set (of nodes, vertices),
- E is a set of non-empty subsets of V (called hyperedges), i.e., $E \subseteq 2^V \setminus \{\emptyset\}$.

Note: Hyperedges can contain arbitrarily many nodes.
Example in the next section.

Feedback Sets

Often, we want to make a graph cycle-free.

Definition (Feedback Arc Set)

Given: A directed graph $G = (V, A)$ and a natural number k.
Question: Is there a subset $A' \subseteq A$ with $|A'| \leq k$ such that A' contains at least one arc from every cycle in G?

Definition (Feedback Vertex Set)

Given: A directed graph $G = (V, A)$ and a natural number k.
Question: Is there a subset $V' \subseteq V$ with $|V'| \leq k$ such that V' contains at least one vertex from every cycle in G?

Similar problems for undirected graphs.

Digraph: Example

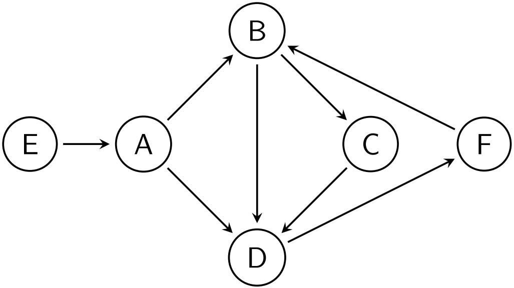

Figure: A directed graph with cycles

Digraph: Example

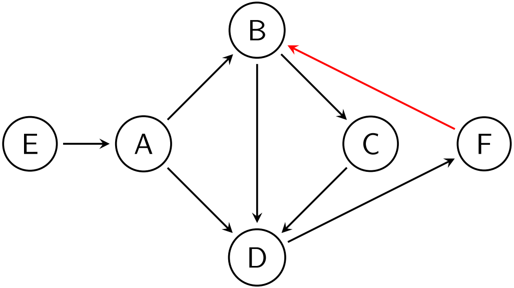

Figure: Feedback arc set

Computational Complexity

Theorem

The following problems are NP-complete:

- *Feedback vertex set for directed graphs,*
- *Feedback arc set for directed graphs,*
- *Feedback vertex set for undirected graphs.*

The feedback edge set for undirected graphs can be solved in polynomial time (maximum spanning tree).

Literature

Rina Dechter.
Constraint Processing,
Chapter 1 and 2, Morgan Kaufmann, 2003

Wikipedia contributors,
Graph theory, Graph (mathematics), Boolean Algebra, Relational Algebra, (2007, April),
In Wikipedia, The Free Encyclopedia. Wikipedia.