Constraint Satisfaction Problems

Mathematical Background: Sets, Relations, and Graphs

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Constraint Satisfaction Problems

October 19 and 21, 2009 — Mathematical Background: Sets, Relations, and Graphs

Sets
- Set-theoretical Principles
- Sets and Boolean Algebras

Relations
- Relations
- Binary Relations
- Relations over Variables

Graphs
- Undirected Graphs
- Directed Graphs
- Hypergraphs
- Graph Problems

Constraints, Sets, Relations, Graphs

- Formal definition of CSP uses sets and constraints
- Constraints are specific relations that restrict possible solutions
- CSP solving techniques use operations that manipulate sets and relations
- CSP instances can also be represented by various kinds of graphs
- Graph-theoretical notions can be used to describe, e.g., structural properties of constraint networks
- Complexity for solving CSP instances can depend on both the relations used in the constraints and properties of the constraint graphs

Sets

Sets:

Naive understanding:

a set is a "well-defined" collection of objects.

Principles/Set-theoretical axioms (ZF):

Axioms that describe which objects count as sets and which operations can be used to form new sets:
extensionality principle, existence of an empty set, pairs and unions of sets, separation principle, power set axioms, axiom of foundations, axiom of replacement, infinte set axiom, axiom of choice, etc.

Set-theoretical Notations:

Boolean operations on sets:

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$
$$A \cap B := \{x \in A : x \in B\}$$
$$A \setminus B := \{x \in A : x \notin B\}$$

Power set: $A \subseteq B$, $A \subsetneq B$, etc., are defined as usual.

$$2^A := \{B : B \subseteq A\}$$

(Ordered) pairs:

$$(x, y) := \{\{x\}, \{x, y\}\}$$
$$(x_1, \ldots, x_n) := ((x_1, \ldots, x_{n-1}), x_n)$$
$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}$$

Boolean Algebra

Definition

A Boolean algebra (with complements) is a set A with

- two binary operations $\sqcap$, $\sqcup$,
- a unary operation $-$, and
- two distinct elements 0 and 1

such that for all elements a, b and c of A:

$$
\begin{array}{ccr}
a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c & a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c & \text{Ass} \\
a \sqcup b = b \sqcup a & a \sqcap b = b \sqcap a & \text{Com} \\
a \sqcup (a \sqcap b) = a & a \sqcap (a \sqcup b) = a & \text{Abs} \\
a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c) & a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c) & \text{Dis} \\
a \sqcup -a = 1 & a \sqcap -a = 0 & \text{Compl}
\end{array}
$$

Sets and Boolean Algebras

Definition

A set algebra on a set X is a non-empty subset of 2^X that is closed under unions, intersections, and complements.

Note: a set algebra on X contains X and $\emptyset$ as elements.

Lemma

(a) *The power set of a set is a set algebra.*

(b) *Each set algebra defines a Boolean algebra.*

Relations

Definition

A relation over sets $X_1, \ldots, X_n$ is a subset

$$R \subseteq X_1 \times \cdots \times X_n =: \prod_{1 \leq i \leq n} X_i.$$

The number n is referred to as arity of R.
An n-ary relation on a set X is a subset

$$R \subseteq X^n := X \times \cdots \times X \quad (n \text{ times}).$$

Since relations are sets, set-theoretical operations (union, intersection, complement) can be applied to relations as well.

Binary Relations

For binary relations on a set X we have some special operations:

Definition

Let R, S be binary (2-ary) relations on X.
The converse of relation R is defined by:

$$R^{-1} := \left\{(x, y) \in X^2 : (y, x) \in R\right\}.$$

The composition of relations R and S is defined by:

$$R \circ S := \left\{(x, z) \in X^2 : \exists y \in X \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S\right\}.$$

The identity relation is:

$$\Delta_X := \left\{(x, y) \in X^2 : x = y\right\}.$$

Operating on Binary Relations

Lemma

Let X be a non-empty set. Let $\mathcal{R}(X)$ be the set of all binary relations on X. Then:

(a) *$\mathcal{R}(X)$ is a set algebra on $X \times X$.*

(b) *For all relations $R, S, T \in \mathcal{R}(X)$:*

$$R \circ (S \circ T) = (R \circ S) \circ T$$

$$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

$$\Delta_X \circ R = R \circ \Delta_X = R$$

$$(R^{-1})^{-1} = R \text{ and } (-R)^{-1} = -(R^{-1})$$

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

$$(R \circ S) \cap T^{-1} = \emptyset \text{ if and only if } (S \circ T) \cap R^{-1} = \emptyset$$

Constraints: Relations over Variables

Let V be a set of variables. For $v \in V$, let $\mathrm{dom}(v)$ be a non-empty set (of values) (the domain of v).

Definition

A relation over (pairwise distinct) variables $v_1, \ldots, v_n \in V$ is an $n+1$-tuple

$$R_{v_1,\ldots,v_n} := (v_1, \ldots, v_n, R)$$

where R is a relation over $\mathrm{dom}(v_1), \ldots, \mathrm{dom}(v_n)$.

The sequence $(v_1, \ldots, v_n)$ is referred to as the range, the set $\{v_1, \ldots, v_n\}$ as the scope, and R as the graph of $R_{v_1,\ldots,v_n}$.

We will not always distinguish between the relation and its graph, e. g., we write

$$R_{v_1,\ldots,v_n} \subseteq \mathrm{dom}(v_1) \times \cdots \times \mathrm{dom}(v_n).$$

Selections, ...

Let $\overline{v} := (v_1, \ldots, v_n)$ and $R_{\overline{v}}$ be a relation over $\overline{v}$.

Definition

For fixed values $a_1 \in \mathrm{dom}(v_{i_1}), \ldots, a_k \in \mathrm{dom}(v_{i_k})$,

$$\sigma_{v_{i_1}=a_1,\ldots,v_{i_k}=a_k}(R_{\overline{v}}) := \left\{(x_1, \ldots, x_n) \in R_{\overline{v}} \;:\; x_{i_j} = a_j, 1 \leq j \leq k\right\}$$

defines a relation over $\overline{v}$.

The (unary) operation $\sigma_{v_{i_1}=a_1,\ldots,v_{i_k}=a_k}$ is called selection or restriction.

… Projections, …

Let $\overline{v} := (v_1, \dots, v_n)$ be as above, and let $(i_1, \dots, i_k)$ be a k-tuple of pairwise distinct elements of $\{1, \dots, n\}$ $(k \leq n)$.
For $\overline{x} = (x_1, \dots, x_n)$, set $\overline{x}_{i_1,\dots,i_k} := (x_{i_1}, \dots, x_{i_k})$.

Definition

For a relation $R_{\overline{v}}$ over $\overline{v}$,

$$\pi_{v_{i_1},\dots,v_{i_k}}(R_{\overline{v}}) := \left\{\overline{y} \in \prod_{1 \leq j \leq k} \mathrm{dom}(v_{i_j}) \ : \ \overline{y} = \overline{x}_{i_1,\dots,i_k}, \text{ for some } \overline{x} \in R_{\overline{v}}\right\}$$

is a relation over $\overline{v}_{i_1,\dots,i_k}$, the projection of $R_{\overline{v}}$ on $\overline{v}_{i_1,\dots,i_k}$.

Note: For binary relations $R = R_{x,y}$, $R^{-1} = \pi_{y,x}(R_{x,y})$.

. . . Joins

For tuples $\overline{x}$ and $\overline{y}$ define:

- $\overline{x} - \overline{y}$: the subsequence of elements in $\overline{x}$ that do not occur in $\overline{y}$.
- $\overline{x} \cap \overline{y}$: the subsequence of $\overline{x}$ with elements that occur in $\overline{y}$.
- $\overline{x} \cup \overline{y}$: the sequence resulting from $\overline{x}$ by adding $\overline{y} - \overline{x}$.

Definition

Let $R_{\overline{v}}$ and $S_{\overline{w}}$ be relations over variables $\overline{v}$ and $\overline{w}$, resp.

$$R_{\overline{v}} \bowtie S_{\overline{w}} := \left\{ \overline{x} \cup \overline{y} \;:\; \overline{x} \in R_{\overline{v}},\; \overline{y} \in R_{\overline{w}},\; \text{and } \overline{x}_{\overline{v} \cap \overline{w}} = \overline{y}_{\overline{v} \cap \overline{w}} \right\}$$

is a relation over $\overline{v} \cup \overline{w}$, the join of $R_{\overline{v}}$ and $S_{\overline{w}}$.

Note: For binary relations $R = R_{x,y}$ and $S = S_{y,z}$ on the same set,

$$R \circ S = \pi_{x,z}(R_{x,y} \bowtie S_{y,z}).$$

Examples

Consider relations $R := R_{x_1,x_2,x_3}$ and $S := S_{x_2,x_3,x_4}$ defined by:

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

Then $\sigma_{x_3=c}(R)$, $\pi_{x_2,x_3}(R)$, $\pi_{x_2,x_1}(R)$, and $R \bowtie S$ are:

x_1	x_2	x_3
b	b	c
c	b	c

x_2	x_3
b	c
b	c
n	n

x_2	x_1
b	b
b	c
n	c

x_1	x_2	x_3	x_4
b	b	c	2
b	b	c	3
c	b	c	2
c	b	c	3

Undirected Graph

Definition

An (undirected, simple) graph is an ordered pair

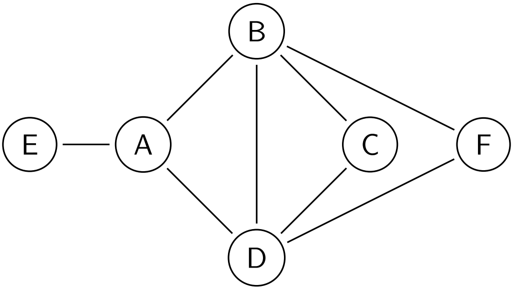
$$G := \langle V, E \rangle$$

where:

- V is a finite set (of vertices, nodes);
- E is a set of two-element subsets of (not necessarily distinct) nodes (called edges).

The order of a graph is the number of vertices $|V|$.
The size of a graph is the number of edges $|E|$.
The degree of a vertex is the number of vertices to which it is connected by an edge.

Graph: Example



Graph: Definitions

Definition

Let $G = \langle V, E \rangle$ be an undirected graph.

(a) If $e = \{u, v\} \in E$, then u and v are called adjacent (or: connected by e).

(b) A path in G is a sequence of vertices $v_0, \ldots, v_k$ such that $\{v_{i-1}, v_i\} \in E$ $(1 \leq i \leq k)$. k is the length, v_0 is the start vertex, and v_k is the end vertex of the path.

(c) A cycle is a path $v_0, \ldots, v_k$ with $v_0 = v_k$.

(d) A path $v_0, \ldots, v_k$ is simple if $v_i \neq v_j$ for all $i \neq j$.

(e) A cycle $v_0, \ldots, v_k$ is simple if $v_i \neq v_j$ for all $i, j \geq 1, i \neq j$.

Graph: Definitions

Let $G = \langle V, E \rangle$ be an undirected graph.

Definition

(a) G is connected if for each pair of vertices u and v, there exists a path from u to v.

(b) G is a tree if G is cycle-free.

(c) G is complete if any pair of vertices is connected by an edge.

Definition

Let S be a subset of V. Then $G_S := \langle S, E_S \rangle$ is called the subgraph relative to S, where $E_S := \big\{\{u, v\} \in E \,:\, u, v \in S\big\}$.

Definition

A clique in a graph G is a complete subgraph of G.

Examples

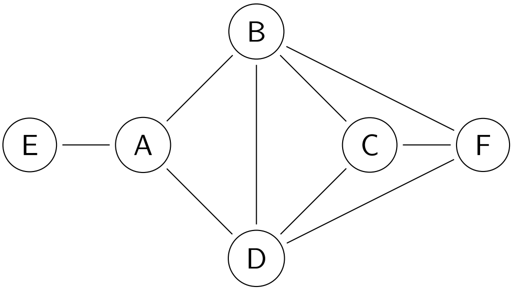


Figure: Example

Directed Graph

Definition

A (simple) directed graph (or: digraph) is an ordered pair

$$G := \langle V, A \rangle$$

where:

- V is a set (of vertices or nodes),
- A is a set of (ordered) pairs of vertices (or: arcs, edges, or arrows).

The number of edges with a vertex v as start vertex is called the outdegree of v; the number of edges with v as end vertex is the indegree of v. Nodes that point to v are called parents, nodes to which an edge from v points are called child nodes.

Digraph: Definitions

Definition

Let $G = \langle V, A \rangle$ be a directed graph.

(a) A (directed) path is a sequence of arcs $e_1, \ldots, e_k$ such that the end vertex of e_i is the start vertex of e_{i+1} (analogously, (directed) cycle).

(b) A digraph is strongly connected if each pair of nodes u, v is connected by a directed path from u to v.

(c) A digraph is acyclic if it has no directed cycles.

Digraph: Example

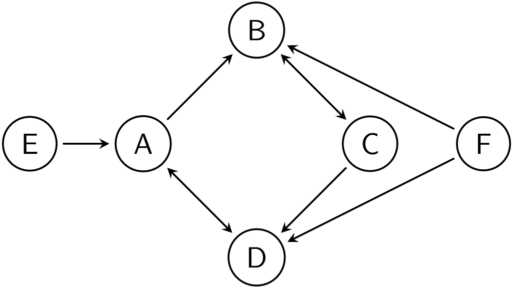


Figure: A directed graph with a strongly connected subgraph

Hypergraph

Graphs can be used to represent binary relations between nodes.
For relations of higher arity we need:

Definition

A hypergraph is a pair

$$H := \langle V, E \rangle$$

where

- V is a set (of nodes, vertices),
- E is a set of non-empty subsets of V (called hyperedges), i.e., $E \subseteq 2^V \setminus \{\emptyset\}$.

Note: Hyperedges can contain arbitrarily many nodes.
Example in the next section.

Feedback Sets

Often, we want to make a graph cycle-free.

Definition (Feedback Arc Set)

Given: A directed graph $G = (V, A)$ and a natural number k.
Question: Is there a subset $A' \subseteq A$ with $|A'| \leq k$ such that A' contains at least one arc from every cycle in G?

Definition (Feedback Vertex Set)

Given: A directed graph $G = (V, A)$ and a natural number k.
Question: Is there a subset $V' \subseteq V$ with $|V'| \leq k$ such that V' contains at least one vertex from every cycle in G?

Similar problems for undirected graphs.

Digraph: Example

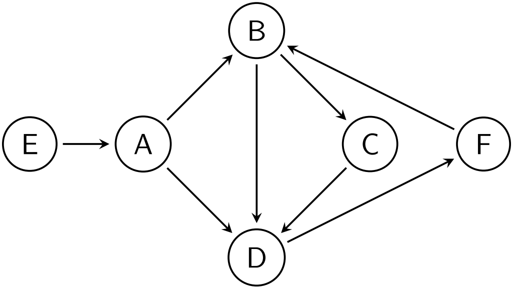


Figure: A directed graph with cycles

Computational Complexity

Theorem

The following problems are NP-complete:

- *Feedback vertex set for directed graphs,*
- *Feedback arc set for directed graphs,*
- *Feedback vertex set for undirected graphs.*

The feedback edge set for undirected graphs can be solved in polynomial time (maximum spanning tree).

Literature

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