

Constraint Satisfaction Problems

Introduction

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based on a slideset by
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Constraint Satisfaction Problems

October 19, 2009 — Introduction

Introduction

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Constraints

What is a constraint?

1 a: the act of constraining **b:** the state of being checked, restricted, or compelled to avoid or perform some action . . . **c:** a constraining condition, agency, or force . . .

2 a: repression of one's own feelings, behavior, or actions **b:** a sense of being constrained . . .

(from *Merriam-Webster's Online Dictionary*)

Usage

- ▶ In programming languages, constraints are often used to restrict the domains of variables.
- ▶ In databases, constraints can be used to specify integrity conditions.
- ▶ In mathematics, a constraint is a requirement on solutions of optimization problems.

Examples

Examples:

- ▶ Latin squares
- ▶ Eight queens problem
- ▶ Sudoku
- ▶ Map coloring problem
- ▶ Boolean satisfiability

Latin Square

Problem:

- ▶ How can one fill an $n \times n$ table with n different symbols
- ▶ ... such that each symbol occurs exactly once in each row and each column?

$$[1] \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

There are essentially 56 different Latin squares of size 5, 9408 squares of size 6, 16.942.080 squares of size 7, 535.281.401.856 squares of size 8, ...

Eight Queens Puzzle

Problem:

- ▶ How can one put 8 queens on a standard chess board (8×8 -board)
- ▶ ... such that no queen can attack any other queen?

Solutions:

- ▶ The puzzle has **12 unique solutions** (up to rotations and reflections)
- ▶ Old problem proposed in 1848.
- ▶ Various variants
 - ▶ knights (instead of queens)
 - ▶ 3D
 - ▶ n queens on an $n \times n$ -board

A Solution ...

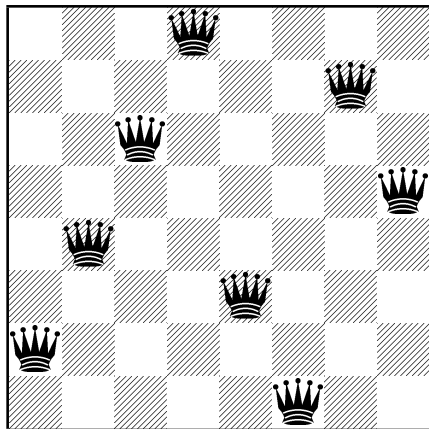


Figure: A solution of the 8-queens problem

Sudoku

Problem:

- ▶ Fill a partially completed 9×9 grid such that
- ▶ ... each row, each column, and each of the nine 3×3 boxes contains the numbers from 1 to 9.

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

Constraint Satisfaction Problem

Definition

A **constraint network** is defined by:

- ▶ a finite set of **variables**
- ▶ a (finite) domain of **values** for each variable
- ▶ a finite set of **constraints** (i.e., binary, ternary, ... relations defined between the variables)

Problem

Is there a **solution** of the network, i.e., an assignment of values to the variables such that all constraints are satisfied?

k -Colorability

Problem:

- ▶ Can one color the nodes of a given graph with k colors
- ▶ ... such that all nodes connected by an edge have different colors?

Reformulated as a constraint network:

- ▶ **Variables:** the nodes of the graph;
- ▶ **Domains:** "colors" $\{1, \dots, k\}$ for each variable;
- ▶ **Constraints:** nodes connected by an edge must have different values.

This constraint network has a particular restricted form:

- ▶ Only **binary** constraints.
- ▶ The domains are **finite**.

Crossword Puzzle

Problem instance:

- ▶ **Variables:** empty squares in a crossword puzzle;
- ▶ **Domains:** letters $\{A, B, C, \dots, Z\}$ for each variable;
- ▶ **Constraints:** relations defined by a given set of words that need (or are allowed) to occur in the completed puzzle.

1	2	3	4	5	6	7	8
9	■	10	■	11	12	■	13
14	15	16	17	18	■	19	20
■	21	22	■	23	24	25	■

Fill-in words: EIER, HOLZ, IE, IM, IT, NZ, ON, RAM, RE,
ROLLE, ROT, ZAR, ZUHOERER

Boolean Satisfiability

Problem instance (Boolean constraint network):

- ▶ **Variables:** (propositional) variables;
- ▶ **Domains:** truth values $\{0, 1\}$ for each variable;
- ▶ **Constraints:** relations defined by a propositional formulae in these variables.

Example: $(x_1 \vee \neg x_2 \vee \neg x_3), (x_1 \vee x_2 \vee x_4)$

SAT as a constraint satisfaction problem:

Given an arbitrary Boolean constraint network, is the network solvable?

3SAT as a constraint satisfaction problem:

Given an arbitrary Boolean constraint network defined by clauses that contain exactly three literals, is the network solvable?

Real World Applications

CSP solving techniques can be used in

- ▶ *civil engineering* (design of power plants, water and energy supply, transportation and traffic infrastructure)
- ▶ *mechanical engineering* (design of machines, robots, vehicles)
- ▶ digital circuit *verification*
- ▶ automated timetabling
- ▶ air traffic control
- ▶ finance

Computational Complexity

Theorem

It is NP-hard to decide solvability of CSPs.

Since k -colorability (SAT, 3SAT) is NP-complete, solvability of CSPs in general must be NP-hard.

Question: Is CSP solvability *in* NP?

Solving CSPs

- ▶ Enumeration of all assignments and testing

↪ ... too costly

- ▶ Backtracking search

↪ numerous different strategies, often "dead" search paths are explored extensively

- ▶ Constraint propagation: elimination of obviously impossible values
- ▶ Interleaving backtracking and constraint propagation: constraint propagation at each generated search node
- ▶ Many other search methods, e. g., local search, stochastic search, etc.

Contents

- ▶ *Introduction and mathematical background*
 - ▶ Sets, relations, graphs
 - ▶ Constraint networks and satisfiability
 - ▶ Binary Constraint Networks
 - ▶ Simple solution methods (backtracking, etc.)
- ▶ *Inference-based methods*
 - ▶ Arc and path consistency
 - ▶ k -consistency and global consistency
- ▶ *Search methods*
 - ▶ Backtracking
 - ▶ Backjumping
 - ▶ Comparing different methods
 - ▶ Stochastic local search
- ▶ *Selected advanced topics*
 - ▶ Expressiveness vs complexity of constraint formalisms
 - ▶ Qualitative constraint networks

Lectures: Where, When, Web Page

Where

101, Room 00-010/14

When

Monday: 11:15–13:00

Wednesday: 11:15–12:00 (+ exercises: 12:15–13:00)

Winter Term Holidays

December 24, 2009 – January 6, 2010

Web Page

<http://www.informatik.uni-freiburg.de/~ki/teaching/ws0910/csp/>

Lecturers

Prof. Bernhard Nebel

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Consultation: Wednesday, 13-14

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Exercises

Who

Robert Mattmüller

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Matthias Westphal

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Where

101, Room 00-010/14

When

Thursday, 15:05–15:50

Examination

Exams

- ▶ Oral exams in February/April 2010

Literature

- ▶ Rina Dechter, *Constraint Processing*, Morgan Kaufmann, 2003.
- ▶ Further readings will be given during the lecture.

Course Prerequisites & Goals

Goals

- ▶ Acquiring skills in constraint processing
- ▶ Understanding the principles behind different solving techniques
- ▶ Being able to read and understand research literature in the area of constraint satisfaction
- ▶ Being able to complete a project (thesis) in this research area

Prerequisites

- ▶ Basic knowledge in the area of AI
- ▶ Basic knowledge in formal logic
- ▶ Basic knowledge in theoretical computer science