Theoretical Computer Science II

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Exercise Sheet 1

Due: October 28, 2009

Exercise 1.1 (Proof by induction, 2+3 marks)

Prove the following two statements *by induction*. Please make clear what is the *base case*, the *induction hypothesis* and the *induction step*.

(a) For $n \in \mathbb{N}^+$:

$$\sum_{i=1}^{n}(2i-1) = n^2$$

(b) For $n \in \mathbb{N}^+$, the power set of the set $S_n = \{1, 2, \ldots, n\}$ contains 2^n elements.

Explanation: The power set $\mathcal{P}(S)$ of a set S is the set of all subsets of S, e.g. $\mathcal{P}(\{1,2\}) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$.

Exercise 1.2 (Direct proof and proof by contradiction, 2+3 marks)

Consider the following definitions of *graph*, *path*, and *tree*.

- A *graph* $G = (V, E)$ is defined by a finite set of vertices (or nodes) V together with a set of edges E where each edge is a 2-element subset of V (a subset $\{v_i, v_j\}$ in E means that there is an edge between node v_i and node v_j).
- A *path* is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. A *simple path* is a path that contains each vertex at most once.
- A *tree* is a *connected* graph that has no *simple cycles*. A graph is connected if each pair of distinct vertices can be connected through some path. A simple cycle in a graph is a path $\langle v_0, v_1, \ldots, v_n \rangle$ with $v_0 = v_n$, $n \geq 3$ and $v_i \neq v_j$ for all $0 \leq i < j < n$.

Consider the following statement:

> Let $G = (V, E)$ be a graph with a vertex v such that there is exactly one simple path from v to each other vertex in V. Then G is a tree.

In order to prove that G is a tree, we have to prove that G is connected and that G does not contain simple cycles.

(a) Prove *directly* that G is connected.

(b) Prove *by contradiction* that G does not contain simple cycles.