Principles of AI Planning

November 8th, 2006 — Planning by state-space search

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Planning by state-space search

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Normal form for effects

1. Similarly to normal forms in propositional logic (DNF, CNF, NNF, ...) we can define a normal form for effects.
2. Nesting of conditionals, as in $a \rhd (b \rhd c)$, can be eliminated.
3. Effects e within a conditional effect $\phi \rhd e$ can be restricted to atomic effects (a or $\neg a$).
4. Only a small polynomial increase in size by transformation to normal form.
   Compare: transformation to CNF or DNF may increase formula size exponentially.

Equivalences on effects

$$
\begin{aligned}
c \rhd (e_1 \land \cdots \land e_n) &\equiv (c \rhd e_1) \land \cdots \land (c \rhd e_n) && (1)\\
c_1 \rhd (c_2 \rhd e) &\equiv (c_1 \land c_2) \rhd e && (2)\\
(c_1 \rhd e) \land (c_2 \rhd e) &\equiv (c_1 \lor c_2) \rhd e && (3)\\
e \land (c \rhd e) &\equiv e && (4)\\
e &\equiv \top \rhd e && (5)\\
e &\equiv \top \land e && (6)\\
e_1 \land e_2 &\equiv e_2 \land e_1 && (7)\\
(e_1 \land e_2) \land e_3 &\equiv e_1 \land (e_2 \land e_3) && (8)
\end{aligned}
$$

Normal form for operators and effects

Definition

An operator $\langle c, e \rangle$ is in normal form if for all occurrences of $c' \rhd e'$ in e the effect e' is either a or $\neg a$ for some $a \in A$, and there is at most one occurrence of any atomic effect in e.

Theorem

For every operator there is an equivalent one in normal form.

Proof is constructive: we can transform any operator into normal form by using the equivalences from the previous slide.

Normal form for effects

Example

Example

$$
\begin{aligned}
&(a \rhd (b \wedge \\
&\qquad (c \rhd (\neg d \wedge e)))) \wedge \\
&(\neg b \rhd e)
\end{aligned}
$$

transformed to normal form is

$$
\begin{aligned}
&(a \rhd b) \wedge \\
&((a \wedge c) \rhd \neg d) \wedge \\
&((\neg b \vee (a \wedge c)) \rhd e)
\end{aligned}
$$

STRIPS operators

Definition

An operator $\langle c, e \rangle$ is a STRIPS operator if

1. c is a conjunction of literals, and
2. e is a conjunction of atomic effects.

Hence every STRIPS operator is of the form

$$\langle l_1 \wedge \cdots \wedge l_n, \quad l'_1 \wedge \cdots \wedge l'_m \rangle$$

where l_i are literals and l'_j are atomic effects.

STRIPS

STanford Research Institute Planning System, Fikes & Nilsson, 1971.

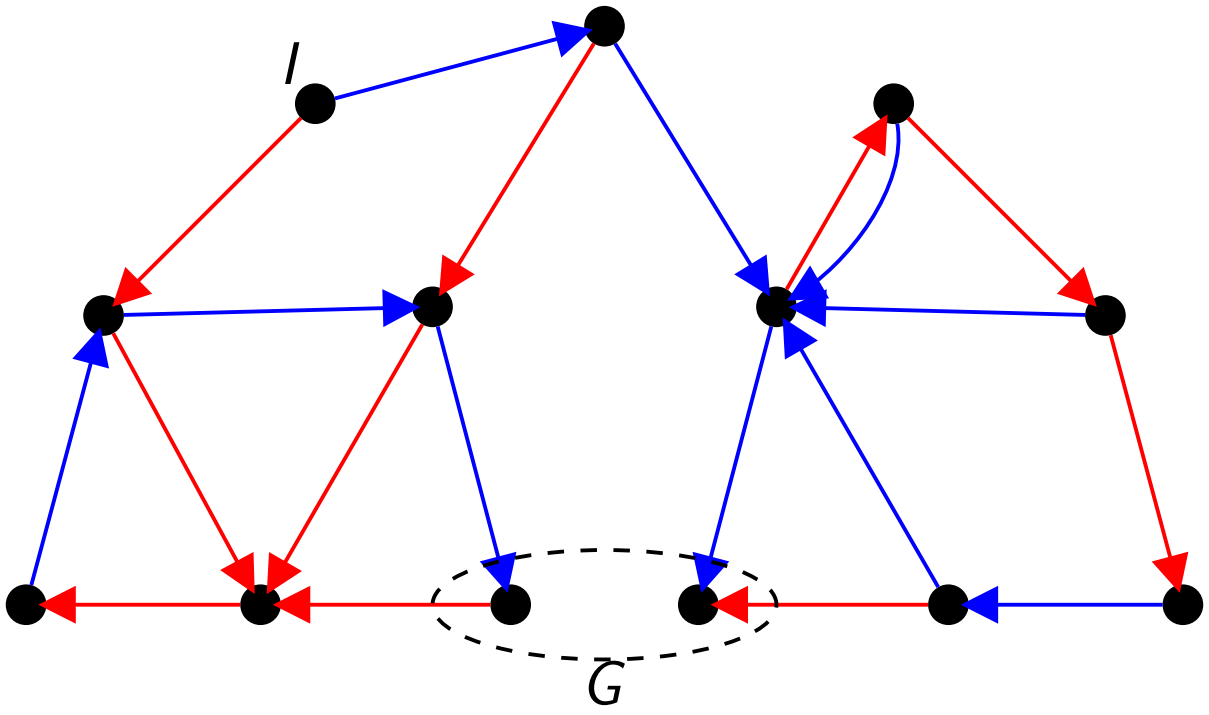
Planning by state-space search

There are many alternative ways of doing planning by state-space search.

1. different ways of expressing planning as a search problem:
   1.1 search direction: forward, backward
   1.2 representation of search space: states, sets of states
2. different search algorithms: depth-first, breadth-first, informed (heuristic) search (systematic: A$*$, IDA$*$, ...; local: hill-climbing, simulated annealing, ...), ...
3. different ways of controlling search:
   3.1 heuristics for heuristic search algorithms
   3.2 pruning techniques: invariants, symmetry elimination, ...

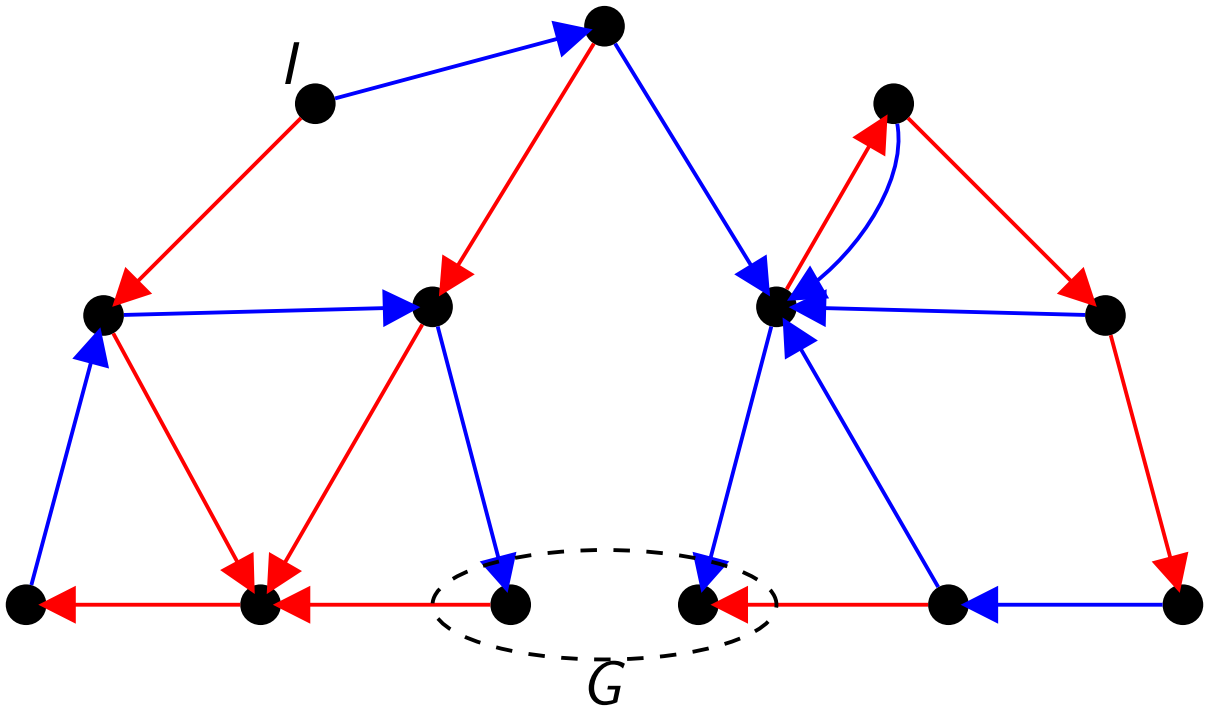
Planning by forward search

with depth-first search



Planning by backward search

with depth-first search, one state at a time

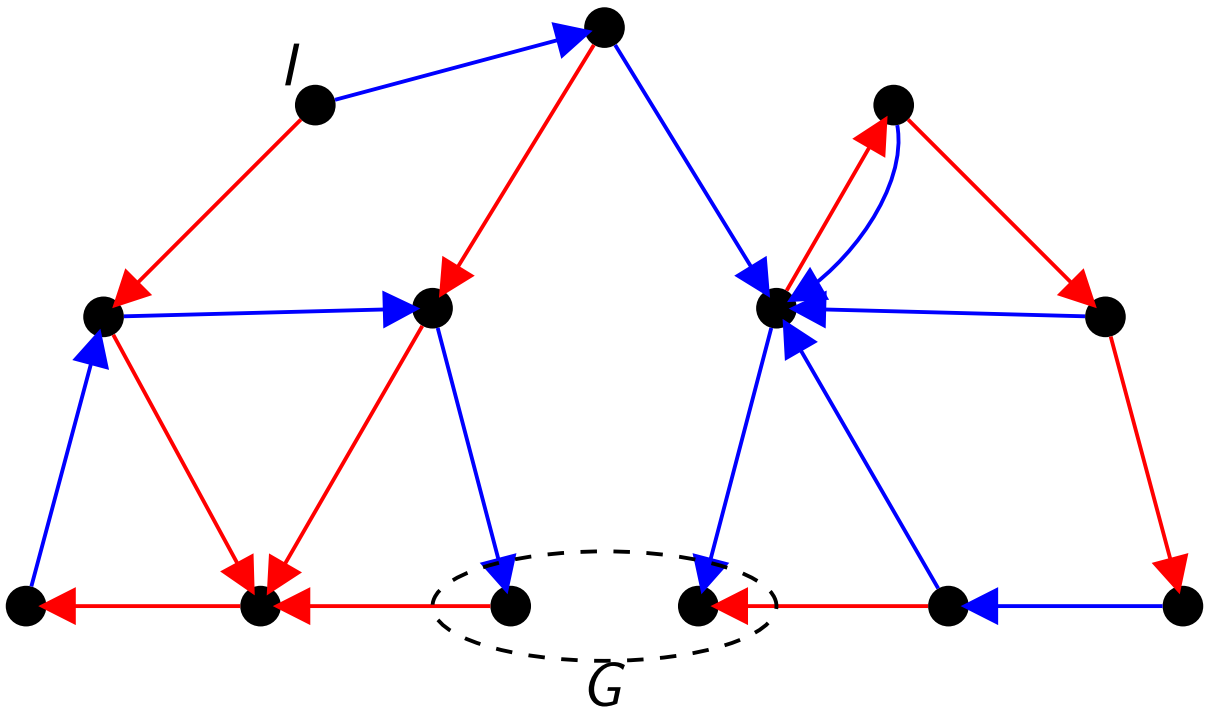


Planning by backward search

with depth-first search, for state sets (represented as formulae)

$$\phi_1 = regr_{\longrightarrow}(G)$$
$$\phi_2 = regr_{\longrightarrow}(\phi_1)$$
$$\phi_3 = regr_{\longrightarrow}(\phi_2), I \models \phi_3$$

$$\phi_3 \longrightarrow \phi_2 \longrightarrow \phi_1 \longrightarrow G$$



Progression

- Progression means computing the successor state $app_o(s)$ of s with respect to o.
- Used in forward search: from the initial state toward the goal states.
- Very easy and efficient to implement.

Regression

- Regression is computing the possible predecessor states of a set of states.
- The formula $regr_o(\phi)$ represents the states from which a state represented by ϕ is reached by operator o.
- Used in backward search: from the goal states toward the initial state.
- Regression is powerful because it allows handling sets of states (progression: only one state at a time.)
- Handling state sets (formulae) is more complicated than handling states: many questions about regression are NP-hard.

Regression for STRIPS operators

- Regression for STRIPS operators is very simple.
- Goals are conjunctions of literals $l_1 \wedge \cdots \wedge l_n$.
- First step: Choose an operator that makes some of $l_1, \ldots, l_n$ true and makes none of them false.
- Second step: Form a new goal by removing the fulfilled goal literals and adding the preconditions of the operator.

Regression for STRIPS operators
Definition

Definition

The STRIPS-regression $regr_o^{str}(\phi)$ of $\phi = l_1'' \wedge \cdots \wedge l_k''$ with respect to

$$o = \langle l_1 \wedge \cdots \wedge l_n, \quad l_1' \wedge \cdots \wedge l_m' \rangle$$

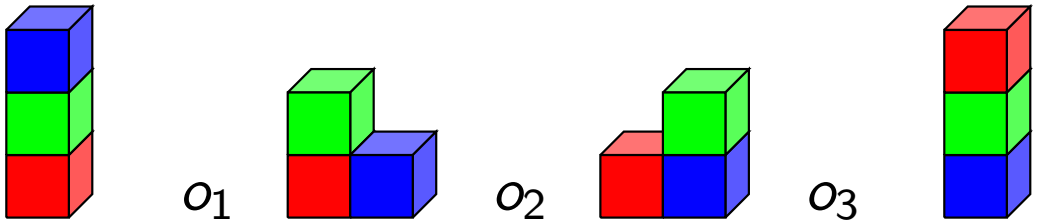
is the conjunction of literals

$$\bigwedge \left((\{l_1'', \ldots, l_k''\} \setminus \{l_1', \ldots, l_m'\}) \cup \{l_1, \ldots, l_n\} \right)$$

provided that $\{l_1', \ldots, l_m'\} \cap \{\overline{l_1''}, \ldots, \overline{l_k''}\} = \emptyset$.

Regression for STRIPS operators

Example



NOTE: Predecessor states are in general not unique.
This picture is just for illustration purposes.

$o_1 = \langle$ ■on■ $\wedge$ ■clr, $\neg$■on■ $\wedge$ ■onT $\wedge$ ■clr$\rangle$

$o_2 = \langle$ ■on■ $\wedge$ ■clr $\wedge$ ■clr, $\neg$■clr $\wedge$ $\neg$■on■ $\wedge$ ■on■ $\wedge$ ■clr$\rangle$

$o_3 = \langle$ ■onT $\wedge$ ■clr $\wedge$ ■clr, $\neg$■clr $\wedge$ $\neg$■onT $\wedge$ ■on■$\rangle$

$G =$ ■on■ $\wedge$ ■on■

$\phi_1 = regr^{str}_{o_3}(G) =$ ■on■ $\wedge$ ■onT $\wedge$ ■clr $\wedge$ ■clr

$\phi_2 = regr^{str}_{o_2}(\phi_1) =$ ■onT $\wedge$ ■clr $\wedge$ ■on■ $\wedge$ ■clr

$\phi_3 = regr^{str}_{o_1}(\phi_2) =$ ■onT $\wedge$ ■on■ $\wedge$ ■clr $\wedge$ ■on■

Regression for general operators

- With disjunction and conditional effects, things become more tricky. How to regress $A \vee (B \wedge C)$ with respect to $\langle Q, D \rhd B \rangle$?
- The story about goals and subgoals and fulfilling subgoals, as in the STRIPS case, is no longer useful.
- We present a general method for doing regression for any formula and any operator.
- Now we extensively use the idea of *representing sets of states as formulae*.

Precondition for effect l to take place: $EPC_l(e)$

Definition

Definition

The condition $EPC_l(e)$ for literal l to become true under effect e is defined as follows.

$$
\begin{aligned}
EPC_l(l) &= \top \\
EPC_l(l') &= \bot \text{ when } l \neq l' \quad \text{(for literals } l'\text{)} \\
EPC_l(\top) &= \bot \\
EPC_l(e_1 \wedge \cdots \wedge e_n) &= EPC_l(e_1) \vee \cdots \vee EPC_l(e_n) \\
EPC_l(c \rhd e) &= EPC_l(e) \wedge c
\end{aligned}
$$

Precondition for effect l to take place: $EPC_l(e)$

Example

Example

$$
\begin{aligned}
EPC_a(b \wedge c) &= \bot \vee \bot \equiv \bot \\
EPC_a(a \wedge (b \rhd a)) &= \top \vee (\top \wedge b) \equiv \top \\
EPC_a((c \rhd a) \wedge (b \rhd a)) &= (\top \wedge c) \vee (\top \wedge b) \equiv c \vee b
\end{aligned}
$$

Precondition for effect l to take place: $EPC_l(e)$

Connection to $[e]_s$

Lemma (A)

Let s be a state, l a literal and e an effect. Then $l \in [e]_s$ if and only if $s \models EPC_l(e)$.

Proof.

Induction on the structure of the effect e.
Base case 1, $e = \top$: By definition of $[\top]_s$ we have $l \notin [\top]_s = \emptyset$ and by definition of $EPC_l(\top)$ we have $s \not\models EPC_l(\top) = \bot$: Both sides of the equivalence are false.
Base case 2, $e = l$: $l \in [l]_s = \{l\}$ by definition, and $s \models EPC_l(l) = \top$ by definition. Both sides are true.
Base case 3, $e = l'$ for some literal $l' \neq l$: $l \notin [l']_s = \{l'\}$ by definition, and $s \not\models EPC_l(l') = \bot$ by definition. Both sides are false.

Precondition for effect l to take place: $EPC_l(e)$

Connection to $[e]_s$

proof continues...

Inductive case 1, $e = e_1 \wedge \cdots \wedge e_n$:

$$
\begin{aligned}
l \in [e]_s \text{ iff } & l \in [e_1]_s \cup \cdots \cup [e_n]_s && \text{(Def } [e_1 \wedge \cdots \wedge e_n]_s\text{)}\\
\text{iff } & l \in [e']_s \text{ for some } e' \in \{e_1, \ldots, e_n\}\\
\text{iff } & s \models EPC_l(e') \text{ for some } e' \in \{e_1, \ldots, e_n\} && \text{(IH)}\\
\text{iff } & s \models EPC_l(e_1) \vee \cdots \vee EPC_l(e_n)\\
\text{iff } & s \models EPC_l(e_1 \wedge \cdots \wedge e_n). && \text{(Def } EPC\text{)}
\end{aligned}
$$

Inductive case 2, $e = c \rhd e'$:

$$
\begin{aligned}
l \in [c \rhd e']_s \text{ iff } & l \in [e']_s \text{ and } s \models c && \text{(Def } [c \rhd e']_s\text{)}\\
\text{iff } & s \models EPC_l(e') \text{ and } s \models c && \text{(IH)}\\
\text{iff } & s \models EPC_l(e') \wedge c\\
\text{iff } & s \models EPC_l(c \rhd e'). && \text{(Def } EPC\text{)}
\end{aligned}
$$

□

Precondition for effect l to take place: $EPC_l(e)$

Connection to the normal form

Remark

Notice that in terms of $EPC_a(e)$ any operator $\langle c, e \rangle$ can be expressed in normal form as

$$\left\langle c, \bigwedge_{a \in A} (EPC_a(e) \rhd a) \wedge (EPC_{\neg a}(e) \rhd \neg a) \right\rangle.$$

Regression: definition for state variables

Regressing a state variable

The formula $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$ expresses the value of $a \in A$ after applying o in terms of values of state variables before applying o: Either

- a became true, or
- a was true before and it did not become false.

Regression: definition for state variables

Example

Let $e = (b \rhd a) \wedge (c \rhd \neg a) \wedge b \wedge \neg d$.

variable	$EPC_{\ldots}(e) \vee (\cdots \wedge \neg EPC_{\neg\ldots}(e))$
a	$b \vee (a \wedge \neg c)$
b	$\top \vee (b \wedge \neg\bot) \equiv \top$
c	$\bot \vee (c \wedge \neg\bot) \equiv c$
d	$\bot \vee (d \wedge \neg\top) \equiv \bot$

Regression: definition for state variables

Lemma (B)

Let a be a state variable, $o = \langle c, e \rangle \in O$ an operator, s a state and $s' = app_o(s)$. Then $s \models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$ if and only if $s' \models a$.

Proof.

First prove the implication from left to right.
Assume $s \models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. Do a case analysis on the two disjuncts.

1. Assume that $s \models EPC_a(e)$. By Lemma A $a \in [e]_s$ and hence $s' \models a$.
2. Assume that $s \models a \wedge \neg EPC_{\neg a}(e)$. By Lemma A $\neg a \notin [e]_s$. Hence a remains true in s'.

Regression: definition for state variables

proof continues...

We showed that if the formula is true in s, then a is true in s'.

For the second part we show that if the formula is false in s, then a is false in s'.

1. So assume $s \not\models EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$.
2. Hence $s \models \neg EPC_a(e) \wedge (\neg a \vee EPC_{\neg a}(e))$ (de Morgan).
3. Analyze the two cases: a is true or it is false in s.
   - 3.1 Assume that $s \models a$. Now $s \models EPC_{\neg a}(e)$ because $s \models \neg a \vee EPC_{\neg a}(e)$. Hence by Lemma A $\neg a \in [e]_s$ and we get $s' \not\models a$.
   - 3.2 Assume that $s \not\models a$. Because $s \models \neg EPC_a(e)$, by Lemma A $a \notin [e]_s$ and hence $s' \not\models a$.

   Therefore in both cases $s' \not\models a$.

$\square$

Regression: general definition

We base the definition of regression on formulae $EPC_l(e)$.

Definition

Let ϕ be a propositional formula and $o = \langle c, e \rangle$ an operator.
The regression of ϕ with respect to o is

$$regr_o(\phi) = c \wedge \phi_r \wedge f$$

where

1. ϕ_r is obtained from ϕ by replacing each $a \in A$ by $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$, and
2. $f = \bigwedge_{a \in A} \neg(EPC_a(e) \wedge EPC_{\neg a}(e))$.

The formula f says that no state variable may become simultaneously true and false.

Regression: examples

1. $regr_{\langle a,b\rangle}(b) \equiv a \wedge (\top \vee (b \wedge \neg\bot)) \wedge \top \equiv a$
2. $regr_{\langle a,b\rangle}(b \wedge c \wedge d) \equiv$
   $a \wedge (\top \vee (b \wedge \neg\bot)) \wedge (\bot \vee (c \wedge \neg\bot)) \wedge (\bot \vee (d \wedge \neg\bot)) \wedge \top \equiv a \wedge c \wedge d$
3. $regr_{\langle a,c\rhd b\rangle}(b) \equiv a \wedge (c \vee (b \wedge \neg\bot)) \wedge \top \equiv a \wedge (c \vee b)$
4. $regr_{\langle a,(c\rhd b)\wedge(b\rhd\neg b)\rangle}(b) \equiv a \wedge (c \vee (b \wedge \neg b)) \wedge \neg(c \wedge b) \equiv a \wedge c \wedge \neg b$
5. $regr_{\langle a,(c\rhd b)\wedge(d\rhd\neg b)\rangle}(b) \equiv a \wedge (c \vee (b \wedge \neg d)) \wedge \neg(c \wedge d) \equiv$
   $a \wedge (c \vee b) \wedge (c \vee \neg d) \wedge (\neg c \vee \neg d)$

Regression: examples

Blocks World with conditional effects

Operators to move blocks A and B onto the table from the other block if they are clear:

$$\begin{aligned} o_1 &= \langle \top, \quad (AonB \wedge Aclear) \rhd (AonT \wedge Bclear \wedge \neg AonB) \rangle \\ o_2 &= \langle \top, \quad (BonA \wedge Bclear) \rhd (BonT \wedge Aclear \wedge \neg BonA) \rangle \end{aligned}$$

Plan for putting both blocks onto the table from any blocks world state with two blocks is o_2, o_1. Proof by regression:

$$\begin{aligned} G &= AonT \wedge BonT \\ \phi_1 &= regr_{o_1}(G) \equiv ((AonB \wedge Aclear) \vee AonT) \wedge BonT \\ \phi_2 &= regr_{o_2}(\phi_1) \\ &\equiv ((AonB \wedge ((BonA \wedge Bclear) \vee Aclear)) \vee AonT) \\ &\quad \wedge ((BonA \wedge Bclear) \vee BonT) \end{aligned}$$

All three 2-block states satisfy ϕ_2. Similar plans exist for any number of blocks.

Regression: examples

Incrementing a binary number

$$(\neg b_0 \rhd b_0) \land$$
$$((\neg b_1 \land b_0) \rhd (b_1 \land \neg b_0)) \land$$
$$((\neg b_2 \land b_1 \land b_0) \rhd (b_2 \land \neg b_1 \land \neg b_0))$$

$$\begin{aligned}
EPC_{b_2}(e) &= \neg b_2 \land b_1 \land b_0 \\
EPC_{b_1}(e) &= \neg b_1 \land b_0 \\
EPC_{b_0}(e) &= \neg b_0 \\
EPC_{\neg b_2}(e) &= \bot \\
EPC_{\neg b_1}(e) &= \neg b_2 \land b_1 \land b_0 \\
EPC_{\neg b_0}(e) &= (\neg b_1 \land b_0) \lor (\neg b_2 \land b_1 \land b_0) \equiv (\neg b_1 \lor \neg b_2) \land b_0
\end{aligned}$$

Regression replaces state variables as follows:

$$\begin{aligned}
b_2 \quad &\text{by} \quad (\neg b_2 \land b_1 \land b_0) \lor (b_2 \land \neg\bot) \equiv (b_1 \land b_0) \lor b_2 \\
b_1 \quad &\text{by} \quad (\neg b_1 \land b_0) \lor (b_1 \land \neg(\neg b_2 \land b_1 \land b_0)) \\
&\qquad \equiv (\neg b_1 \land b_0) \lor (b_1 \land (b_2 \lor \neg b_0)) \\
b_0 \quad &\text{by} \quad \neg b_0 \lor (b_0 \land \neg((\neg b_1 \lor \neg b_2) \land b_0)) \equiv \neg b_0 \lor (b_1 \land b_2)
\end{aligned}$$

Regression: properties

Lemma (C)

Let ϕ be a formula, o an operator, s any state and $s' = app_o(s)$. Then $s \models regr_o(\phi)$ if and only if $s' \models \phi$.

Proof.

Let e be the effect of o. We show by structural induction over subformulae ϕ' of ϕ that $s \models \phi'_r$ iff $s' \models \phi'$, where ϕ'_r is ϕ' with every $a \in A$ replaced by $EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$. Rest of $regr_o(\phi)$ just states that o is applicable in s.

Induction hypothesis $s \models \phi'_r$ if and only if $s' \models \phi'$.

Base cases 1 & 2 $\phi' = \top$ or $\phi' = \bot$: Trivial as $\phi'_r = \phi'$.

Base case 3 $\phi' = a$ for some $a \in A$: Now
$\phi'_r = EPC_a(e) \vee (a \wedge \neg EPC_{\neg a}(e))$.
By Lemma B $s \models \phi'_r$ iff $s' \models \phi'$.

Regression: properties

proof continues...

Inductive case 1 $\phi' = \neg\psi$: By the induction hypothesis $s \models \psi_r$ iff $s' \models \psi$. Hence $s \models \phi'_r$ iff $s' \models \phi'$ by the truth-definition of $\neg$.

Inductive case 2 $\phi' = \psi \vee \psi'$: By the induction hypothesis $s \models \psi_r$ iff $s' \models \psi$, and $s \models \psi'_r$ iff $s' \models \psi'$. Hence $s \models \phi'_r$ iff $s' \models \phi'$ by the truth-definition of $\vee$.

Inductive case 3 $\phi' = \psi \wedge \psi'$: By the induction hypothesis $s \models \psi_r$ iff $s' \models \psi$, and $s \models \psi'_r$ iff $s' \models \psi'$. Hence $s \models \phi'_r$ iff $s' \models \phi'$ by the truth-definition of $\wedge$.

$\square$

Regression: complexity issues

The following two tests are useful when generating a search tree with regression.

1. Testing that a formula $regr_o(\phi)$ does not represent the empty set (= search is in a blind alley).
   For example, $regr_{\langle a,\neg p\rangle}(p) \equiv a \wedge \bot \equiv \bot$.
2. Testing that a regression step does not make the set of states smaller (= more difficult to reach).
   For example, $regr_{\langle b,c\rangle}(a) \equiv a \wedge b$.

Both of these problems are NP-hard.

Regression: complexity issues

The formula $regr_{o_1}(regr_{o_2}(\ldots regr_{o_{n-1}}(regr_{o_n}(\phi))))$ may have size $\mathcal{O}(|\phi||o_1||o_2|\ldots|o_{n-1}||o_n|)$, i.e. the product of the sizes of ϕ and the operators.
The size in the worst case $\mathcal{O}(m^n)$ is hence exponential in n.

Logical simplifications

1. $\bot \wedge \phi \equiv \bot$, $\top \wedge \phi \equiv \phi$, $\bot \vee \phi \equiv \phi$, $\top \vee \phi \equiv \top$
2. $a \vee \phi \equiv a \vee \phi[\bot/a]$, $\neg a \vee \phi \equiv \neg a \vee \phi[\top/a]$, $a \wedge \phi \equiv a \wedge \phi[\top/a]$, $\neg a \wedge \phi \equiv \neg a \wedge \phi[\bot/a]$

To obtain the maximum benefit from the last equivalences, e.g. for $(a \wedge b) \wedge \phi(a)$, the equivalences for associativity and commutativity are useful: $(\phi_1 \vee \phi_2) \vee \phi_3 \equiv \phi_1 \vee (\phi_2 \vee \phi_3)$, $\phi_1 \vee \phi_2 \equiv \phi_2 \vee \phi_1$, $(\phi_1 \wedge \phi_2) \wedge \phi_3 \equiv \phi_1 \wedge (\phi_2 \wedge \phi_3)$, $\phi_1 \wedge \phi_2 \equiv \phi_2 \wedge \phi_1$.

Regression: generation of search trees

Problem Formulae obtained with regression may become very big.

Cause Disjunctivity in the formulae. Formulae without disjunctions easily convertible to small formulae $l_1 \wedge \cdots \wedge l_n$ where l_i are literals and n is at most the number of state variables.

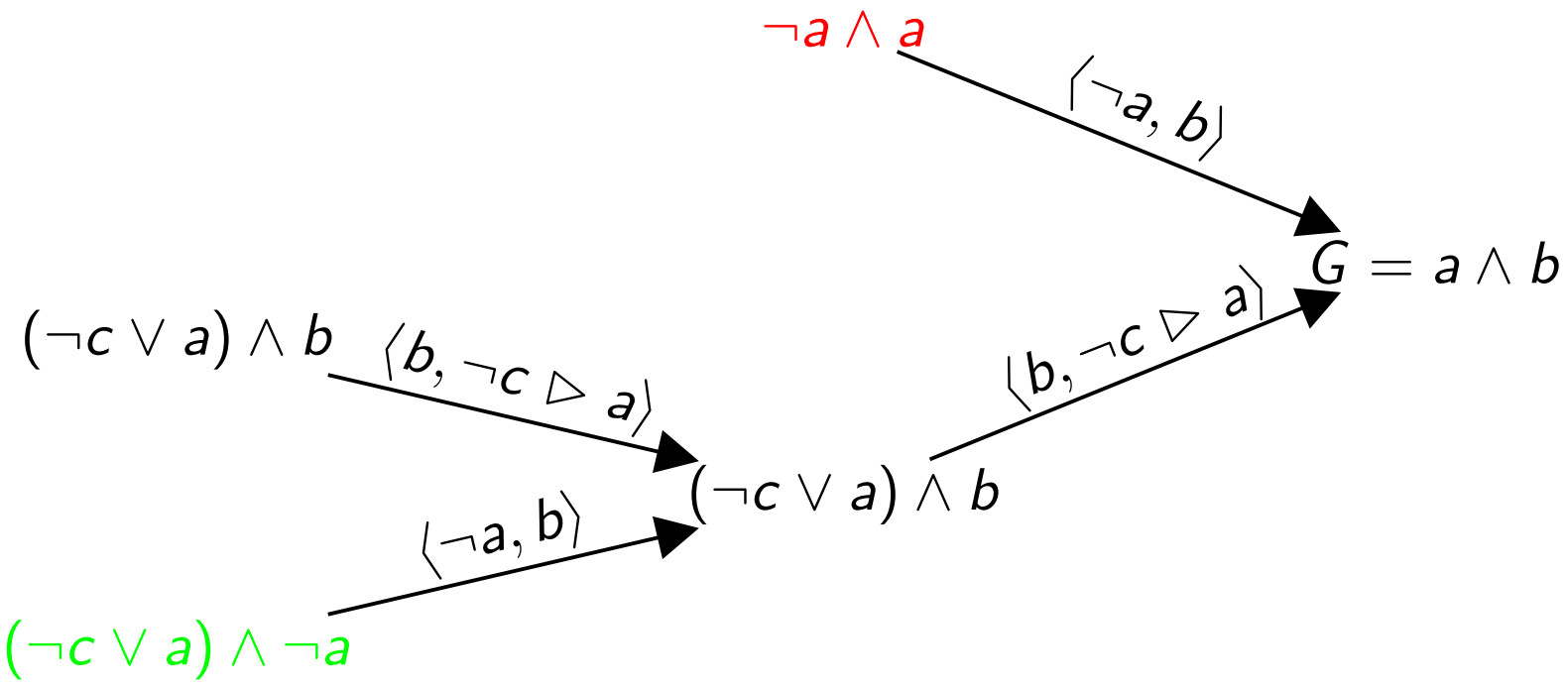
Solution Handle disjunctivity when generating search trees. Alternatives:

1. Do nothing. (May lead to very big formulae!!!)
2. Always eliminate all disjunctivity.
3. Reduce disjunctivity if formula becomes too big.

Regression: generation of search trees

Unrestricted regression (= do nothing about formula size)

Reach goal $a \wedge b$ from state I such that $I \models \neg a \wedge \neg b \wedge \neg c$.



Regression: generation of search trees

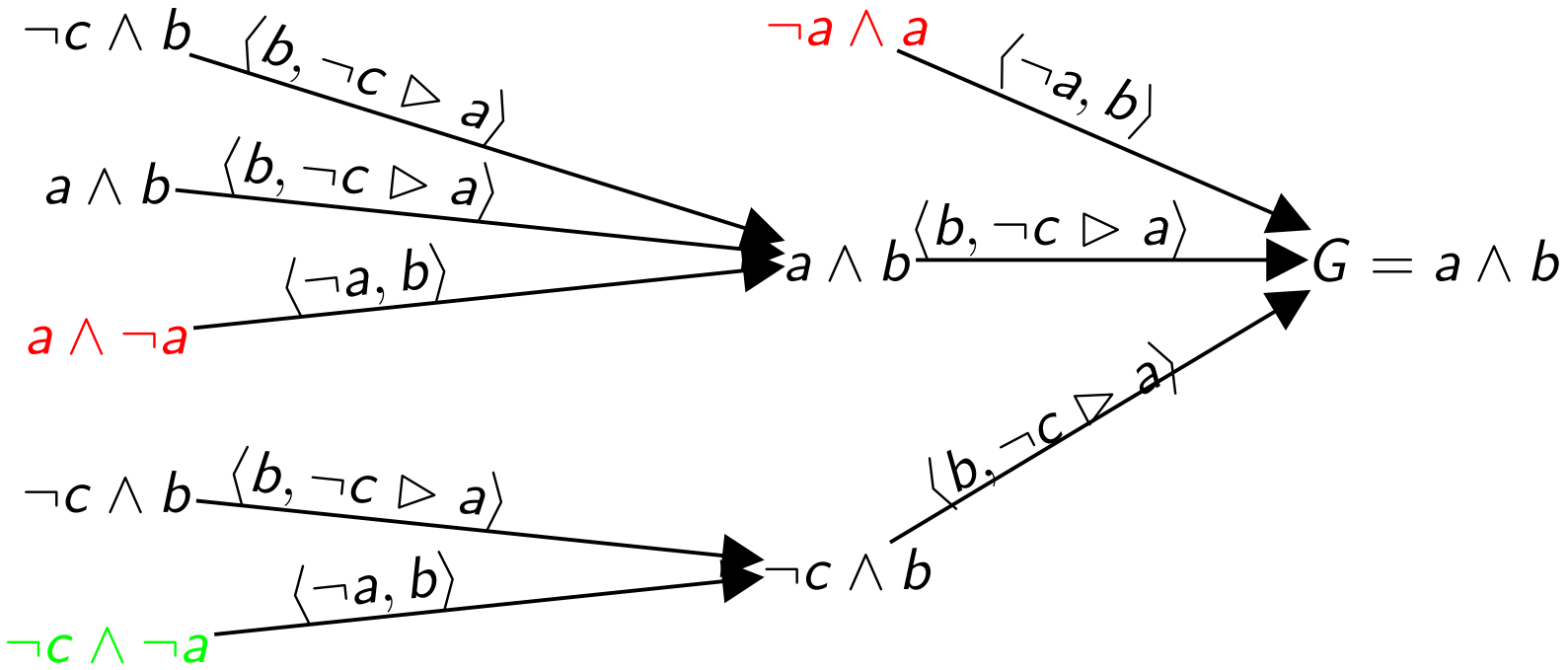
Full splitting (= eliminate all disjunctivity)

- Planners for STRIPS operators only need to use formulae $l_1 \wedge \cdots \wedge l_n$ where l_i are literals.
- Some PDDL planners also restrict to this class of formulae. This is done as follows.
  1. $regr_o(\phi)$ is transformed to disjunctive normal form (DNF): $(l_1^1 \wedge \cdots \wedge l_{n_1}^1) \vee \cdots \vee (l_1^m \wedge \cdots \wedge l_{n_m}^m)$.
  2. Each disjunct $l_1^i \wedge \cdots \wedge l_{n_i}^i$ is handled in its own subtree of the search tree.
  3. The DNF formulae need not exist in its entirety explicitly: generate one disjunct at a time.
- Hence branching is both on the choice of operator and on the choice of the disjunct of the DNF formula.
- This leads to an increased branching factor and bigger search trees, but avoids big formulae.

Regression: generation of search trees

Full splitting

Reach goal $a \wedge b$ from state I such that $I \models \neg a \wedge \neg b \wedge \neg c$.
$(\neg c \vee a) \wedge b$ in DNF is $(\neg c \wedge b) \vee (a \wedge b)$.
It is split to $\neg c \wedge b$ and $a \wedge b$.



Regression: generation of search trees

Restricted splitting

- With full splitting search tree can be exponentially bigger than without splitting. (But it is not necessary to construct the DNF formulae explicitly!)
- Without splitting the formulae may have size that is exponential in the number of state variables.
- A compromise is to split formulae only when necessary: combine benefits of the two extremes.
- There are several ways to split a formula ϕ to $\phi_1, \ldots, \phi_n$ such that $\phi \equiv \phi_1 \vee \cdots \vee \phi_n$. For example:
  1. Transform ϕ to $\phi_1 \vee \cdots \vee \phi_n$ by equivalences like distributivity $(\phi_1 \vee \phi_2) \wedge \phi_3 \equiv (\phi_1 \wedge \phi_3) \vee (\phi_2 \wedge \phi_3)$.
  2. Choose state variable a, set $\phi_1 = a \wedge \phi$ and $\phi_2 = \neg a \wedge \phi$, and simplify with equivalences like $a \wedge \psi \equiv a \wedge \psi[\top/a]$.