# An Introduction to Game Theory Part III: Strictly Competitive Games

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# Strictly Competitive Games

- A *strictly competitive* or zero-sum game is a 2-player strategic game such that for each $a \in A$, we have $u_1(a) + u_2(a) = 0$.
  - What is good for me, is bad for my opponent and *vice versa*
- Note: Any game where the sum is a constant $c$ can be transformed into a zero-sum game with the same set of equilibria:
  - $u'_1(a) = u_1(a)$
  - $u'_2(a) = u_2(a) - c$

# How to Play Zero-Sum Games?

- Assume that only *pure strategies* are allowed
- ➢ Dominating strategy?
- ➢ Nash equilibrium?
- Be paranoid: Try to minimize your loss by assuming the worst!
- Player 1 takes minimum over row values:
  - T: -6, M: -1, B: -6
- then maximizes:
  - M: -1

| | L | M | R |
|---|---|---|---|
| T | 8,-8 | 3,-3 | -6,6 |
| M | 2,-2 | -1,1 | 3,-3 |
| B | -6,6 | 4,-4 | 8,-8 |

# Maximinimizer

- An action $x^*$ is called *maximinimizer* for player 1, if

  $$\min_{y \in A_2} u_1(x^*,y) \geq \min_{y \in A_2} u_1(x,y) \quad \text{for all } x \in A_1$$

- Similar for player 2
- Maximinimizer try to minimize the loss, but do not necessarily lead to a Nash equilibirium.
- However, if a NE exists, then the action profile is a pair of maximinimizers!

# Maximinimizer Theorem

In strictly competitive games:

1. If $(x^*,y^*)$ is a Nash equilibrium of $G$ then $x^*$ is a maximinimizer for player 1 and $y^*$ is a maximinimizer for player 2.
2. If $(x^*,y^*)$ is a Nash equilibrium of $G$ then $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y) = u_1(x^*,y^*)$.
3. If $\max_x \min_y u_1(x,y) = \min_y \max_x u_1(x,y)$ and $x^*$ is a maximinimizer for player 1 and $y^*$ is a maximinimizer for player 2, then $(x^*, y^*)$ is a Nash equilibrium.

# Some Consequences

- Because of (2): if $(x^*,y^*)$ is a NE then $\max_x \min_y u_1(x,y) = u_1(x^*,y^*)$, all NE yield the same payoff
  - it is irrelevant which we choose.
- Because of (2), if $(x^*,y^*)$ and $(x', y')$ are a NEs then $x^*$, $x'$ are maximinimizers for player 1 and $y^*$, $y'$ are maximinimizers for player 2. Because of (3), then $(x^*,y')$ and $(x',y^*)$ are NEs as well!
  - it is not necessary to coordinate in order to play in a NE!

# Example

- Minimum in rows (for player 1):
  - T: -6, M: -1, B: -6
- Maximinimizer:
  - M: -1
- Maximum over columns (for player 1)
  - L: 8, M: -1, R: 8
- Minimaximizer:
  - M: -1
- Also NE, apparently

| | L | M | R |
|---|---|---|---|
| T | 8,-8 | -3,3 | -6,6 |
| M | 2,-2 | -1,1 | 3,-3 |
| B | -6,6 | -4,4 | 8,-8 |

# How to Find NEs in Mixed Strategies?

- While it is non-trivial to find NEs for general sum games, zero-sum games are "easy"
- Let's test all mixed strategies of player 1 $\alpha_1$ against all mixed strategies of player 2 $\alpha_2$. Then use only those that are maximinimizers.
- Since all mixed strategies are linear combinations of pure strategies, it is enough to check against the pure strategies of player 2 (support theorem).
- We just have to optimize, i.e., find the best mixed strategy
  - Use linear programming

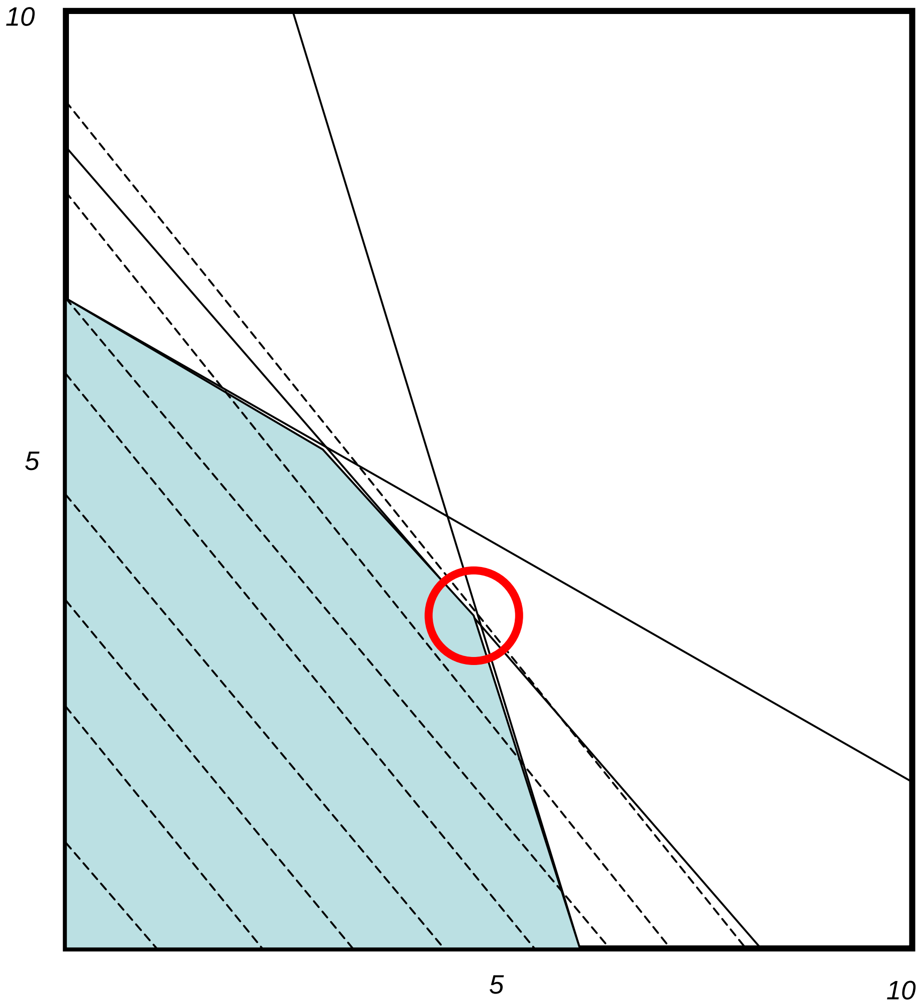
# Linear Programming: The Idea

- The article-mix problem:
  - *article 1* needs: 25 min of cutting, 60 min of assembly, 68 min of postprocessing
    - results in 30 Euro profit per article
  - *article 2* needs: 75 min of cutting, 60 min of assembly, and 34 min of postprocessing
    - results in 40 Euro profit per article
  - *per day*: 450 min of cutting, 480 min of assembly and 476 min of postprocessing
- Try to maximize profit

# Resulting Constraints & Optimization Goals

- $x$: #article1, $y$: #article2
- $x \geq 0, y \geq 0$
- $25x+75y \leq 450$ (cutting)
  - ➢ $y \leq 6-(1/3 \cdot x)$
- $60x+60y \leq 480$ (assembly)
  - ➢ $y \leq 8 - x$
- $68x+34y \leq 476$ (postprocessing)
  - ➢ $y \leq 14 - 2x$
- **Maximize** $z = 30x+40y$

# Feasible Solutions




- The inequalities describe convex sets in $R^2$
- The intersection of all convex sets represents the set of feasible solutions
- Each point in the set of feasible solutions could get a quality measure according to the objective function
- Consider lines of equal quality and then do hill climbing!

# Linear Programming: The Standard Form

- *n* real-valued variables $x_i \geq 0$
- *n* coefficients $b_i$ and m constants $c_j$
- $m \cdot n$ coefficients $a_{ij}$
- *m* equations $\sum_i a_{ij} x_i = c_j$
- objective function: $\sum_i b_i x_i$ is to be minimized
- Can be solved by the simplex method
  - *lpsolve* for example

# Other Forms

- Maximization instead of minimization:
  - set $b'_i = - b_i$
- Inequalities
  - introduce slack (non-negative) variables $z_l$:
  - $\sum_i a_{ij} x_i \leq c_j$ *iff* $\sum_i a_{ij} x_i + z_l = c_j$
- Larger or equal
  - Multiply both sides with -1

# Solving Zero-Sum Games

- Let $A_1 = \{a_{11}, \ldots, a_{1n}\}$, $A_2 = \{a_{21}, \ldots, a_{2m}\}$,
- Player 1 looks for a mixed strategy $\alpha_1$
  - $\sum_j \alpha_1(a_{1j}) = 1$
  - $\alpha_1(a_{1j}) \geq 0$
  - $\sum_j \alpha_1(a_{1j}) \cdot u_1(a_{1j}, a_{2i}) \geq u$ for all $i \in \{1, \ldots, m\}$
  - Maximize *u!*

- Similarly for player 2.

# Conclusion

- Zero-sum games are particularly simple
- Playing a pure maximinizing strategy minimizes loss (for pure strategies)
- If NE exists, it is a pair of maximinimizers
- NEs can be freely "mixed"
- In mixed strategies, NEs always exists
- Can be determined by linear programming