An Introduction to Game Theory Part II: Mixed and Correlated Strategies

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Randomizing Actions …

- Since there does not seem to exist a rational decision, it might be best to randomize strategies.
- Play Head with probability *p* and Tail with probability *1-p*
- Switch to expected utilities

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Some Notation

- Let $G = (N, (A_i), (u_i))$ be a strategic game
- Then $\Delta(A_i)$ shall be the set of probability distributions over A_i – the set of mixed strategies $\alpha_i \in \Delta(A_i)$
- $\alpha_i(a_i)$ is the probability that a_i will be chosen in the mixed strategy α_i
- A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on A: $p(a) = \Pi_i \, \alpha_i(a_i)$
- The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) \, u_i(a)$

Example of a Mixed Strategy

- Let
  - $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$
  - $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$
- Then
  - $p(H,H) = 2/9$
  - …
  - $U_1(\alpha_1, \alpha_2) = ?$
  - $U_2(\alpha_1, \alpha_2) = ?$

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Mixed Extensions

- The mixed extension of the strategic game $(N, (A_i), (u_i))$ is the strategic game $(N, \Delta(A_i), (U_i))$.
- The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.
- Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.

Nash's Theorem

Theorem. Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is finite
- So, there exists always a solution
- What is the computational complexity?
- This is an open problem! Not known to be NP-hard, but there is no known polynomial time algorithm
- Identifying a NE with a value larger than a particular value is NP-hard

The Support

- We call all pure actions a_i that are chosen with non-zero probability by α_i the support of the mixed strategy α_i

Lemma. Given a finite strategic game**,** α^* is a *mixed strategy equilibrium* if and only if for every player *i every pure strategy in the support* of α_i^* is a best response to α_{-i}^* .

Proving the Support Lemma

→ Assume that α^* is a Nash equilibrium with a_i being in the support of α_i^* but not being a best response to α_{-i}^*.

- This means, by reassigning the probability of a_i to the other actions in the support, one can get a higher payoff for player *i.*
- This implies α^* is not a Nash equilibrium □ contradiction

← (Proving the contraposition): Assume that α^* is not a Nash equilibrium.

- This means that there exists α_i' that is a better response than α_i^* to α_{-i}^*.
- Then because of how U_i is computed, there must be an action a_i' in the support of α_i' that is a better response (higher utility) to α_{-i}^* than a pure action a_i^* in the support of α_i^*.
- This implies that there are actions in the support of α_i^* that are not best responses to α_{-i}^*.

Using the Support Lemma

- The Support Lemma can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
- There are 4 potential Nash equilibria in pure strategies
  - *Easy to check*
- There are another 4 potential Nash equilibrium types with a 1-support (pure) against 2-support mixed strategies
  - Exists only if the corresponding pure strategy profiles are already Nash equilibria (follows from Support Lemma)
- There exists one other potential Nash equilibrium type with a 2-support against a 2-support mixed strategies
  - Here we can use the Support Lemma to compute an NE (if there exists one)

1-Support Against 2-Support

	L	R
T	5,5	5,5
B	-100,6	6,1

- There is one NE in pure strategies: (T,L)
- There are many mixed NEs of type $\alpha_1(T) = 1$ and $\alpha_2(L), \alpha_2(R) > 0$
- It is clear that one of L or R must form a NE together with T!

- Assume mixed NE with first strategy (T) of player one as pure strategy:
  - $U_1((1,0), (\alpha_2(L), \alpha_2(R))) \geq U_1((0,1), (\alpha_2(L), \alpha_2(R)))$
  - $u_1(T,L)\alpha_2(L) + u_1(T,R)\alpha_2(R) \geq u_1(B,L)\alpha_2(L) + u_1(B,R)\alpha_2(R)$
- Because of this inequation, it follows that either:
  - $u_1(T,L) \geq u_1(B,L)$ or
  - $u_1(T,R) \geq u_1(B,R)$
- Since it is NE, it is clear that
  - $u_2(T,L) = u_2(T,R)$
- Hence, either T,L or T,R must be a NE

A Mixed Nash Equilibrium for Matching Pennies

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- There is clearly no NE in pure strategies
- Lets try whether there is a NE α^* in mixed strategies
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy α_{-1}^*

- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$
- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$
- $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H)+1\alpha_2(T)$
- $\alpha_2(H)-\alpha_2(T)=-\alpha_2(H)+\alpha_2(T)$
- $2\alpha_2(H) = 2\alpha_2(T)$
- $\alpha_2(H) = \alpha_2(T)$
- Because of $\alpha_2(H)+\alpha_2(T) = 1$:

➢ $\alpha_2(H)=\alpha_2(T)=1/2$

➢ Similarly for player 1!

❖ $U_1(\alpha^*) = 0$

Mixed NE for BoS

	Bach	Stra-vinsky
Bach	2,1	0,0
Stra-vinsky	0,0	1,2

- There are obviously 2 NEs in pure strategies
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

- $U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S)))$
- $U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B)+0\alpha_2(S)$
- $U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B)+1\alpha_2(S)$
- $2\alpha_2(B) = 1\alpha_2(S)$
- Because of $\alpha_2(B)+\alpha_2(S) = 1$:

➢ $\alpha_2(B)=1/3$

➢ $\alpha_2(S)=2/3$

➢ Similarly for player 1!

❖ $U_1(\alpha^*) = 2/3$

Couldn't we Help the BoS Players?

- BoS have two pure strategy Nash equilibria
  - but which should they play?
- They can play a mixed strategy, but this is worse than any pure strategy
- The solution is to talk about, where to go
- Use an external random signal to decide where to go

➢ Correlated Nash equilibria

➢ In the BoS case, we get a payoff of 1.5

The 2/3 of Average Game

- You have *n* players that are allowed to choose a number between 1 and *K*.
- The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners
- What number would you play?
- What mixed strategy would you play?

➢ Are there NEs in pure and/or mixed strategies?

➢ **Let's play it:** Please write down a number between 1 and 100.

A Nash Equilibrium in Pure Strategies

- All playing 1 is a NE in pure strategies
  - A deviation does not make sense
- All playing the same number different from 1 is not a NE
  - Choosing the number just below gives you more
- Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.
- So: ***Why did you not choose 1?***
- Perhaps you acted rationally by assuming that the others do not act rationally?

Are there Proper Mixed Strategy Nash Equilibria?

- Assume there exists a mixed NE α different from the pure NE (1,1,…,1)
- Then there exists a maximal $k^* > 1$ which is played by some player with a probability > 0.
  - Assume player i does so, i.e., k^* is in the support of α_i.
- This implies $U_i(k^*,\alpha_{-i}) > 0$, since k^* should be as good as all the other strategies of the support.
- Let a be a realization of α s.t. $u_i(a) > 0.$ Then at least one other player must play $k^*,$ because not all others could play below 2/3 of the average!
- In this situation player i could get more by playing k^*-1.
- This means, playing k^*-1 is better than playing $k^*,$ i.e., k^* cannot be in the support, i.e., α cannot be a NE

Conclusion

- Although Nash equilibria do not always exist, one can give a guarantee, when we randomize finite games:
- ➢ For every finite strategic game, there exists a Nash equilibrium in mixed strategies
- Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff Support Lemma
- The Support Lemma can be used to determine mixed strategy NEs for 2-person games with 2x2 action sets
- In general, there is no poly-time algorithm known for finding one Nash equilibrium (and identifying one with a given strictly positive payoff is NP-hard)
- In addition to pure and mixed NEs, there exists the notion of correlated NE, where you coordinate your action using an external randomized signal