An Introduction to Game Theory Part I: Strategic Games

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Strategic Game

- A strategic game G consists of
 - a finite set N (the set of players)
 - for each player $i \in N$ a non-empty set A_i (the set of actions or strategies available to player i), whereby $A = \Pi_i A_i$
 - for each player i ∈ N a function $u_i : A → R$ (the utility or payoff function)
 - $-G = (N, (A_i), (u_i))$
- If A is finite, then we say that the game is *finite*

Playing the Game

- Each player i makes a decision which action to play: a_i
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, ..., a_n)$
- Then each player gets the payoff u_i(a*)
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- Note: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
 - If we want to model something like this, the payoff function must be changed

Notation

- For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns
- The payoff for every action profile is specified as a pair x,y, whereby x is the value for player 1 and y is the value for player 2
- Example: For (T,R), player 1 gets x₁₂, and player 2 gets y₁₂

	Player 2 L action	Player 2 R action
Player1 T action	<i>X</i> ₁₁ , <i>Y</i> ₁₁	X ₁₂ , y ₁₂
Player1 B action	<i>X</i> ₂₁ , <i>Y</i> ₂₁	X ₂₂ , y ₂₂

Example Game: Bach and Stravinsky

- Two people want to out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the Battle of the Sexes

	Bach	Stra- vinsky
Bach	2,1	0,0
Stra- vinsky	0,0	1,2

Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
- This game is also called chicken.

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Example Game: Prisoner's Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

	Don't confess	Confess
Don't confess	3,3	0,4
Confess	4,0	1,1

Solving a Game

- What is the right move?
- Different possible solution concepts
 - Elimination of strictly or weakly dominated strategies
 - Maximin strategies (for minimizing the loss in zerosum games)
 - Nash equilibrium
- How difficult is it to compute a solution?
- Are there always solutions?
- Are the solutions unique?

Strictly Dominated Strategies

Notation.

- Let $a = (a_i)$ be a strategy profile
- $-a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
- $-(a_{-i}, a'_{i}) := (a_{1}, \ldots, a_{i-1}, a'_{i}, a_{i+1}, \ldots a_{n})$
- Strictly dominated strategy:
 - An strategy a_j^* ∈ A_j is *strictly dominated* if there exists a strategy a_i^* such that for all strategy profiles $a \in A$:

$$u_{i}(a_{-i}, a_{i}') > u_{i}(a_{-i}, a_{i}')$$

 Of course, it is not rational to play strictly dominated strategies

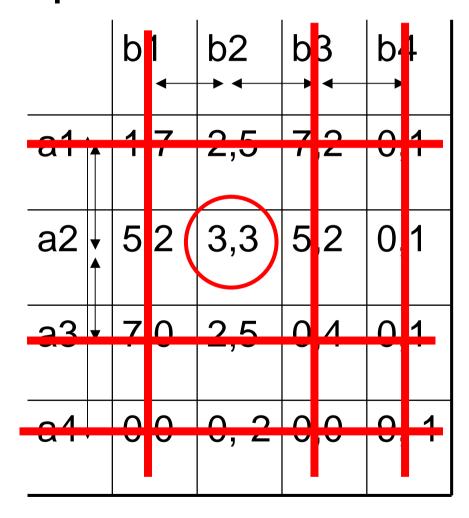
Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game
- This can be done iteratively
- If this converges to a single strategy profile, the result is unique
- This can be regarded as the result of the game, because it is the only rational outcome

Iterated Elimination: Example

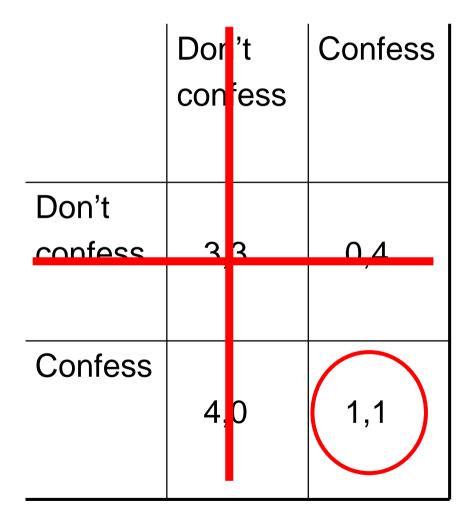
• Eliminate:

- b4, dominated by b3
- a4, dominated by a1
- b3, dominated by b2
- a1, dominated by a2
- b1, dominated by b2
- a3, dominated by a2
- ➤ Result: (a2,b2)



Iterated Elimination: Prisoner's Dilemma

- Player 1 reasons that "not confessing" is strictly dominated and eliminates this option
- Player 2 reasons that player 1 will not consider "not confessing". So he will eliminate this option for himself as well
- So, they both confess



Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
 - An strategy $a_j^* \in A_j$ is weakly dominated if there exists a strategy a_j^* such that for all strategy profiles $a \in A$:

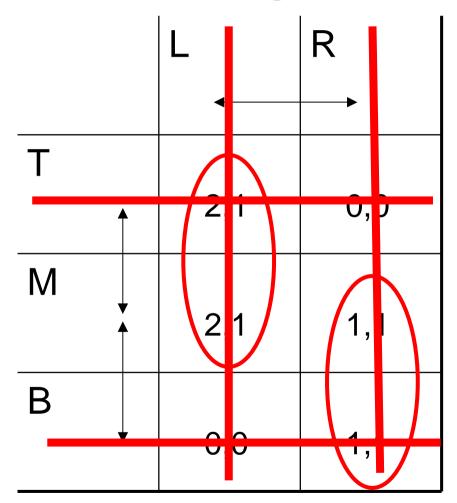
$$u_{j}(a_{-j}, a_{j}') \geq u_{j}(a_{-j}, a_{j}')$$

and for at least one profile $a \in A$:

$$u_i(a_{-i}, a_i') > u_i(a_{-i}, a_i^*).$$

Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
 - Eliminate
 - T (≤M)
 - L (≤R)
 - ➤ Result: (1,1)
 - Eliminate:
 - B (≤M)
 - R (≤L)
 - ➤ Result (2,1)



Analysis of the Guessing 2/3 of the Average Game

- All strategies above 67 are weakly dominated, since they will never ever lead to winning the prize, so they can be eliminated!
- This means, that all strategies above

2/3 x 67

can be eliminated

- ... and so on
- ... until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!

Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?
- Nash equilibrium

	Dove	Hawk
Dove	3,3	1,4
Hawk	4,1	0,0

Nash Equilibrium

- A Nash equilibrium is an action profile a* ∈ A with the property that for all players i ∈ N:
 - $u_i(a^*) = u_i(a^*_{-i}, a^*_i) \ge u_i(a^*_{-i}, a_i) \ \forall \ a_i \in A_i$
- In words, it is an action profile such that there is no incentive for any agent to deviate from it
- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept
- If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium

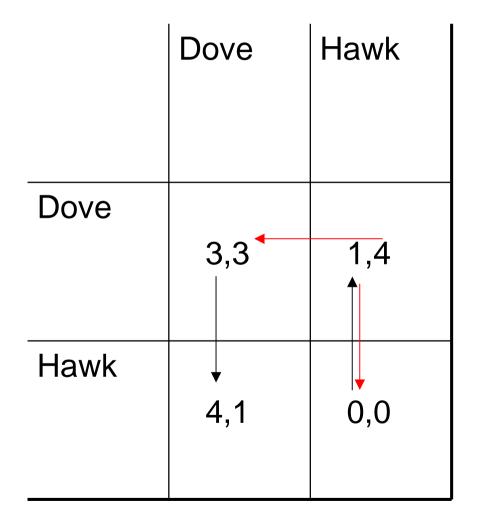
Example Nash-Equilibrium: Prisoner's Dilemma

- Don't Don't
 - not a NE
- Don't Confess (and vice versa)
 - not a NE
- Confess Confess
 - NE

	Don't confess	Confess
Don't confess	3,3	0,4
Confess	4,0	1,1

Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:
 - not a NE
- Hawk-Hawk
 - not a NE
- Dove-Hawk
 - is a NE
- Hawk-Dove
 - is, of course, anotherNE
- So, NEs are not necessarily unique



Auctions

- An object is to be assigned to a player in the set {1,...,n} in exchange for a payment.
- Players *i* valuation of the object is v_i , and $v_1 > v_2 > ... > v_n$.
- The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (nonnegative real numbers)
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
- The payment for a first price auction is the highest bid.
- What are the Nash equilibria in this case?

Formalization

- Game G = $(\{1,...,n\}, (A_i), (u_i))$
- A_i : bids $b_i \in \mathbb{R}^+$
- $u_i(b_{-i}, b_i) = v_i b_i$ if *i* has won the auction, 0 othwerwise
- Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.

Nash Equilibria for First-Price Sealed-Bid Auctions

- The Nash equilibria of this game are all profiles b with:
 - $-b_i \le b_1$ for all $i \in \{2, ..., n\}$
 - No i would bid more than v₂ because it could lead to negative utility
 - If a b_i (with $< v_2$) is higher than b_1 player 1 could increase its utility by bidding $v_2 + \varepsilon$
 - So 1 wins in all NEs
 - $v_1 \ge b_1 \ge v_2$
 - Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility
 - $-b_i = b_1$ for at least one $j \in \{2, ..., n\}$
 - Otherwise player 1 could have gotten the object for a lower bid

Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game
- No NE at all! What shall we do here?

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

Conclusions

- Strategic games are one-shot games, where everybody plays its move simultaneously
- The game outcome is the action profile resulting from the individual choices.
- Each player gets a payoff based on its payoff function and the resulting action profile.
- Iterated elimination of strictly dominated strategies is a convincing solution concept, but unfortunately, most of the time it does not yield a unique solution
- Nash equilibrium is another solution concept: Action profiles, where no player has an incentive to deviate
- It also might not be unique and there can be even infinitely many NEs.
- Also, there is no guarantee for the existence of a NE