Qualitative Representation and Reasoning

Introduction

Knowledge Representation and Reasoning

January 16, 2006

Outline

Quantitative vs. Qualitative

Spatio-temporal configurations can be described quantitatively by specifying the coordinates of the relevant objects:

Example: *At time point 10.0 object A is at position $(11.0, 1.0, 23.7)$, at time point 11.0 at position $(15.2, 3.5, 23.7)$. From time point 0.0 to 11.0, object B is at position $(15.2, 3.5, 23.7)$. Object C is at time point 11.0 at position $(300.9, 25.6, 200.0)$ and at time point 35.0 at $(11.0, 1.0, 23.7)$.*

Often, however, a qualitative description (using a finite vocabulary) is more adequate:

Example: *Object A hit object B. Afterwards, object C arrived.*

Sometimes we want to reason with such descriptions, e.g.:

Object C was not close to object A when it hit object B.

Representation of Qualitative Knowledge

Intention: Description of configurations using a finite vocabulary and reasoning about these descriptions

- Specification of a vocabulary: usually a finite set of relations (often binary) that are pairwise disjoint and exhaustive
- Specification of a language: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- Specification of a formal semantics
- Analysis of computational properties and design of reasoning methods (often constraint propagation)
- Perhaps, specification of operational semantics for verifying whether a relation holds in a given quantitative configuration

Applications in . . .

- Natural language processing
- Specification of abstract spatio-temporal configurations
- Query languages for spatio-temporal information systems
- Layout descriptions of documents (and learning of such layouts)
- Action planning
- . . .

Qualitative Temporal Relations: *Point Calculus*

We want to talk about time instants (points) and binary relations over them.

- *Vocabulary*:
  - X equals Y: $X = Y$
  - X before Y: $X < Y$
  - X after Y: $X > Y$
- *Language*:
  - Allow for disjunctions of basic relations to express indefinite information. Use set of relations to express that. For instance, $\{<, =\}$ expresses $\leq$.
  - 2^3 different relations (including the *impossible* and the *universal* relation)
  - Use sets of atomic formulae with these relations to describe configurations. For example:

$$\{x\{=\}y, y\{<,>\}z\}$$

- *Semantics*: Interpret the time point symbols and relation symbols over the rational (or real) numbers.

Some Reasoning Problems

$$\left\{x\{<,=\}y, y\{<,=\}z, v\{<,=\}y, w\{>\}y, z\{<,=\}x\right\}$$

- Satisfiability: Are there values for all time points such that all formulae are satisfied?
- Satisfiability with $v\{=\}w$?
- Finding a satisfying instantiation of all time points
- Deduction: Does $x\{=\}y$ logically follow?
  Does $v\{<,=\}w$ follow?
- Finding a minimal description: What are the most constrained relations that describe the same set of instantiations?

From a Logical Point of View . . .

In general, qualitatively described configurations are simple logical theories:

- Only sets of atomic formulae to describe the configuration
- Only existentially quantified variables (or constants)
- A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- We are interested in satisfiability, model finding, and *deduction*
- *Constraint Satisfaction Problems*

CSP – Definition

Definition

A *constraint satisfaction problem (CSP)* is given by

- a set V of n variables $\{v_1, \ldots, v_n\}$,
- for each v_i, a value domain D_i
- constraints (relations over subsets of the variables)

Tasks:

Find one (or all) *solution(s)*, i. e., tuples

$$(d_1, \ldots, d_n) \in D_1 \times \ldots \times D_n$$

such that the assignment $v_i \mapsto d_i$ $(1 \leq i \leq n)$ satisfies all constraints.

CSP – Example

k-colorability: Can we color the nodes of a graph with k colors in a way such that all nodes connected by an edge have different colors?

- The node set is the set of variables
- The domain of each variable is $\{1, \ldots, k\}$
- The constraints are that nodes connected by an edge must have a different value

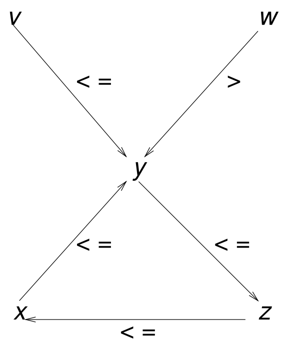
Note: This CSP has a particular restricted form:

- Only binary constraints
- The domains are finite

Other examples: Many problems (e.g. cross-word puzzle, n-queens problem, configuration, ...) can be cast as a CSP (and solved this way)

Our Example: Point relations

- Our point relation CSP is a binary CSP with infinite domains.
- It can be represented as a *constraint graph*:



Computational Complexity

Theorem

It is NP-hard to decide solvability of CSPs, even binary CSPs.

Proof.

Since k-colorability is NP-complete (even for fixed $k \geq 3$), solvability of CSPs in general must be NP-hard. □

Question: Is CSP solvability *in* NP?

Solving CSP

- Enumeration of all assignments and testing
- ⇝ ... too costly
- Backtracking search
- ⇝ 1001 different strategies, often "dead" search paths are explored extensively
- Constraint propagation: elimination of obviously impossible values followed by backtracking search
- Many other search methods, e.g., local search, stochastic search, etc.
- ⇝ How do we solve CSP with infinite domains?

General Assumptions

- Only at most binary constraints (i.e., we can use constraint graph)
- Uniform domain D for all variables
- Unary constraints D_i and binary constraints R_{ij} are sets of values or sets of pairs of values, resp.
- We assume that for all nodes i, j:

$$(x,y) \in R_{ij} \Rightarrow (y,x) \in R_{ji}$$

Local Consistency

- A CSP is *locally consistent* if for particular subsets of the variables, solutions of the restricted CSP can be extended to solutions of a larger set of variables.
- ⇝ Methods to transform a CSP into a tighter, but "equivalent" problem.

Definition

A binary CSP $\langle V,D,C\rangle$ is *arc consistent* (or *2-consistent*) if for all nodes $1 \leq i,j \leq n$,

$$x \in D_i \Rightarrow \exists y \in D_j \text{ s.t. } (x,y) \in R_{ij}$$

- ⇝ When a CSP is arc consistent, each one variable assignment $\{v_i\} \rightarrow D$ that satisfies all (unary) constraints in v_i, i. e., D_i, can be extended to a two variable assignment $\{v_i, v_j\} \rightarrow D$ that satisfies all unary/binary constraints in these variables, i. e., D_i, D_j, and R_{ij}.

Arc Consistency

EnforceArcConsistency $(\mathcal{C})$:
Input: a (binary) CSP $\mathcal{C} = \langle V,D,C\rangle$
Output: an equivalent, but arc consistent CSP $\mathcal{C}'$

repeat
  for each arc (v_i, v_j) with $R_{ij} \in C$
    $D_i := D_i \cap \{x \in D : \text{ex. } y \in D_j \text{ s.t. } (x,y) \in R_{ij}\}$
  endfor
until no domain is changed

- Terminates in time $O(n^3 \cdot k^3)$ if we have finite domains (where k is the number of values)
- ⇝ There exist different (more efficient) algorithms for enforcing arc consistency.

Arc Consistency

Lemma

- *Enforcing arc consistency yields an arc consistent CSP.*
- *Enforcing arc consistency is solution invariant, i. e. it does not change the set of solutions.*

⇝ Arc consistent CSPs need not be consistent, and vice versa.

Arc Consistency – Example

$$\begin{aligned} D_1 &= \{1,2,3\} \\ D_2 &= \{2,3\} \\ D_3 &= \{2\} \\ R_{ij} &= \text{"}\neq\text{" for } i \neq j \end{aligned}$$

1. $D_1 := D_1 \cap \{x : y \in D_3 \wedge (x,y) \in R_{13}\} = \{1,3\}$
2. $D_2 := D_2 \cap \{x : y \in D_3 \wedge (x,y) \in R_{23}\} = \{3\}$
3. $D_1 := D_1 \cap \{x : y \in D_2 \wedge (x,y) \in R_{12}\} = \{1\}$
4. CSP is now arc consistent

- Since all unary constraints are singletons, this defines a solution of the CSP.
- Since enforcing arc consistency does not change the set of solutions, this is a unique solution of the original CSP.

Local Consistency (2): Path Consistency

Definition

A binary CSP $\langle V,D,C\rangle$ is said to be *path consistent* (or *3-consistent*) if for all nodes $1 \leq i,j,k \leq n$,

$$x \in D_i, y \in D_j, (x,y) \in R_{ij} \Rightarrow \exists z \in D_k \text{ s. t. } (x,z) \in R_{ik} \text{ and } (y,z) \in R_{jk}$$

⇝ When a CSP is path consistent, each two variable assignment $\{v_i, v_j\} \to D$ satisfying all constraints in v_i and v_j can be extended to any three variable assignment $\{v_i, v_j, v_k\} \to D$ such that all constraints in these variables are satisfied.

Path Consistency

EnforcePathConsistency $(\mathcal{C})$:
Input: a (binary) CSP $\mathcal{C} = \langle V,D,C\rangle$ of size n
Output: an equivalent, but path consistent CSP $\mathcal{C}'$

repeat
  for all $1 \leq i,j,k \leq n$
    $R_{ij} := R_{ij} \cap$
      $\{(x,y) : \text{ex. } z \in D_k \text{ s. t. } (x,z) \in R_{ik} \text{ and } (y,z) \in R_{jk}\}$
  endfor
until no binary constraint is changed

⇝ Terminates in time $O(n^5 \cdot k^5)$ if we have finite domains (where k is the number of values)

⇝ Enforcing path consistency is solution invariant.

Local Consistency (3): k-Consistency and Strong k-Consistency

Definition

- A binary CSP $\langle V, D, C\rangle$ is *k-consistent* if, given variables $x_1, \ldots, x_k$ and an assignment $a : \{x_1, \ldots, x_{k-1}\} \rightarrow D$ that satisfies all constraint in these variables, a can be extended to an assignment $a' : \{x_1, \ldots, x_k\} \rightarrow D$ that satisfies all constraints in these k variables.
- A binary CSP $\langle V, D, C\rangle$ is *strongly k-consistent* if it is k'-consistent for each $k' \leq k$.
- A binary CSP $\langle V, D, C\rangle$ is *globally consistent* if it is strongly n-consistent where n is the size of V.

Local Consistency (3)

- k-consistency: The computation costs grow exponentially with k.
- If a CSP is globally consistent, then
  - a solution can be constructed in polynomial time,
  - its constraints are minimal,
  - and it has a solution iff there is no empty constraint.
- k-consistent $\not\Rightarrow$ $k-1$-consistent

Qualitative Reasoning with CSP

If we want to use CSPs for qualitative reasoning, we have

- infinite domains
- mostly only finitely many relations (basic relations and their unions)
- arc consistent CSPs (usually)

Questions:

- How do we achieve k-consistency (for some fixed k)?
- Is k-consistency (for some fixed k) enough to guarantee global consistency?
- Is a CSP with only base relations always satisfiable?

Operations on Binary Relations

Composition:
$$R_1 \circ R_2 = \{(x,y) \in D^2 : \exists z \in D \text{ s.t. } (x,z) \in R_1 \text{ and } (z,y) \in R_2\}$$

Converse:
$$R^{-1} = \{(x,y) \in D^2 : (y,x) \in R\}$$

Intersection:
$$R_1 \cap R_2 = \{(x,y) \in D^2 : (x,y) \in R_1 \text{ and } (x,y) \in R_2\}$$

Union:
$$R_1 \cup R_2 = \{(x,y) \in D^2 : (x,y) \in R_1 \text{ or } (x,y) \in R_2\}$$

Complement:
$$\overline{R} = \{(x,y) \in D^2 : (x,y) \notin R\}$$

Conditions on Vocabulary for Qualitative Reasoning

- Let $\mathbf{B}$ be a finite set of (binary) base relations.
- $\rightsquigarrow$ The relations in $\mathbf{B}$ should be JEPD, i. e., jointly exhaustive and pairwise disjoint.
- $\mathbf{B}$ should be *closed under converse*.
- Let $\mathbf{A}$ be the set of relations that can be built by taking the unions of relations from $\mathbf{B}$ ($\rightsquigarrow 2^{|\mathbf{B}|}$ different relations).
- $\rightsquigarrow$ $\mathbf{A}$ is closed under converse, complement, intersection and union.
- $\mathbf{A}$ should be *closed under composition of base relations*, i. e., for all $B, B' \in \mathbf{B}$, $B \circ B' \in A$.
- $\rightsquigarrow$ $\mathbf{A}$ is closed under composition of arbitrary relations.
- $\rightsquigarrow$ This condition does not hold necessarily.
  Example: $\mathbf{B} = \{<,=,>\}$ interpreted over the integers is not closed under composition (and has no finite closure):

$$< \circ < \; = \; < \setminus \{(i,j) : i = j-1\} \subsetneq <$$

Computing Operations on Relations

Let $\mathbf{A}$ be a relation system over the set of base relations $\mathbf{B}$ that satisfies the conditions spelled out above.

$\rightsquigarrow$ We may write relations as *sets* of base relations:

$$B_1 \cup \cdots \cup B_n \sim \{B_1, \ldots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \ldots B_n\} \circ \{B'_1, \ldots, B'_m\} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_i \circ B'_j$$

Converse:

$$\{B_1, \ldots, B_n\}^{-1} = \{B_1^{-1}, \ldots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \ldots, B_n\}} = \{B \in \mathbf{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and union are defined set-theoretically.

Reasoning Problems

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- Given in addition xRy: Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- Compute for each pair x, y the strongest constrained (minimal) relation entailed by the CSP.

$\rightsquigarrow$ These problems are equivalent under Turing reductions

Reductions between CSP Problems

Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.

CSAT $\leq_T$ CENT and CENT $\leq_T$ CMIN are obvious.

CENT $\leq_T$ CSAT: We solve CENT ($CSP \models xRy$?) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let $B_1, \ldots, B_k$ be the relations for which we get a positive answer. Then $x\{B_1, \ldots, B_k\}y$ is entailed by the CSP.

CMIN $\leq_T$ CENT: We use entailment for computing the minimal constraint for each pair. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

Path Consistency for Qualitative CSPs

Given a qualitative CSP with $R_{ij} = R_{ji}^{-1}$. Then path consistency can be enforced by doing the following:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}).$$

Path consistency guarantees ...

- sometimes minimality
- sometimes satisfiability
- however sometimes the CSP is not satisfiable, even if the CSP contains only base relations

$\rightsquigarrow$ All this depends on the vocabulary.

Example: Point Relations

Composition table:

	$<$	$=$	$>$
$<$	$<$	$<$	$<,=,>$
$=$	$<$	$=$	$>$
$>$	$<,=,>$	$>$	$>$

Figure: Composition table for the point algebra. For example: $\{<\} \circ \{=\} = \{<\}$

- $\{<,=\} \circ \{<\} = \{<\}$
- $\{<,>\} \circ \{<\} = \{<,=,>\}$
- $\{<,=\}^{-1} = \{>,=\}$
- $\{<,=\} \cap \{>,=\} = \{=\}$

Some Properties of the Point Relations

Theorem

A path consistent CSP over the point relations is consistent.

Corollary

CSAT, CENT and CMIN are polynomial problems for the point relations.

Theorem

A path consistent CSP over all point relations without $\{<,>\}$ is minimal.

Proofs later ...

A Pathological Relation System

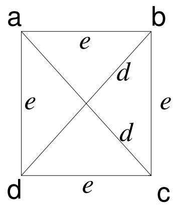
Let e, d, i be (self-converse) base relations between points on a circle:

- e: Rotation by 72 degrees (left or right)
- d: Rotation by 144 degrees (left or right)
- i: Identity

Composition table:

$$e \circ e = \{i, d\}$$
$$d \circ d = \{i, e\}$$
$$e \circ d = \{e, d\}$$
$$d \circ e = \{e, d\}$$

The following CSP is path consistent and contains only base relations, but it is not satisfiable:



Outlook

- Qualitative representation and reasoning usually starts with a finite vocabulary (a finite set of relations).
- Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- Reasoning problems are (as usual) satisfiability, model finding, and deduction.
- Can be addressed with CSP methods (but note: infinite domains).
- Path consistency is the basic reasoning step . . . sometimes this is enough.
- Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.
- Can be taken further $\rightsquigarrow$ relation algebra

Literature I

Alan K. Mackworth.
Constraint satisfaction.
In S. C. Shapiro, editor, *Encyclopedia of Artificial Intelligence*, pages 205–211. Wiley, Chichester, England, 1987.

Alan K. Mackworth.
Consistency in networks of relations.
Artificial Intelligence, 8:99–118, 1977.

Peter B. Ladkin and Roger Maddux.
On binary constraint networks.
Journal of the ACM, 41:435–469, 1994.

Ugo Montanari.
Networks of constraints: fundamental properties and applications to picture processing.
Information Science, 7:95–132, 1974.

R. Hirsch.
Tractable approximations for temporal constraint handling.
Artificial Intelligence, 116: 287-295, 2000.
(Contains the pathological set of relations.)