Semantic Networks and Description Logics

Description Logics – Algorithms

Knowledge Representation and Reasoning

December 14, 2005

Description Logics – Algorithms

Reasoning Problems & Algorithms

- *Satisfiability* or *subsumption* of concept descriptions
- *Satisfiability* or *instance relation* in ABoxes
- **Structural subsumption algorithms**
  - *Normalization* of concept descriptions and *structural comparison*
  - very fast, but can only be used for small DLs
- **Tableau algorithms**
  - Similar to modal tableau methods
  - Meanwhile the method of choice

Structural Subsumption Algorithms

- **Small Logic $\mathcal{FL}^-$**
  - $C \sqcap D$
  - $\forall r.C$
  - $\exists r$ (simple existential quantification)
- **Idea**
  1. In the conjunction, collect all *universally quantified expressions* (also called *value restrictions*) with the same role and build *complex value restriction*:

     $$\forall r.C \sqcap \forall r.D \;\to\; \forall r.(C \sqcap D).$$

  2. Compare all conjuncts with each other. For each conjunct in the subsuming concept there should be a *corresponding one* in the subsumed one.

Example

$$D = \texttt{Human} \sqcap \exists\texttt{has-child} \sqcap \forall\texttt{has-child.Human} \sqcap \forall\texttt{has-child.}\exists\texttt{has-child}$$
$$C = \texttt{Human} \sqcap \texttt{Female} \sqcap \exists\texttt{has-child} \sqcap \forall\texttt{has-child.}(\texttt{Human} \sqcap \texttt{Female} \sqcap \exists\texttt{has-child})$$

Check: $C \sqsubseteq D$

1. Collect value restrictions in D: $\ldots\forall\texttt{has-child.}(\texttt{Human} \sqcap \exists\texttt{has-child})$
2. Compare:
   - 2.1 For Human in D, we have Human in C
   - 2.2 For $\exists$has-child in D, we have ...
   - 2.3 For $\forall$has-child.(...) in D, we have ...
     - 2.3.1 For Human ...
     - 2.3.2 For $\exists$has-child ...

$\rightsquigarrow$ C is subsumed by D!

Subsumption Algorithm

SUB(C,D) algorithm:

1. Reorder terms (*commutativity*, *associativity* and *value restriction law*):

$$C = \bigsqcap A_i \sqcap \bigsqcap \exists r_j \sqcap \bigsqcap \forall r_k : C_k$$
$$D = \bigsqcap B_l \sqcap \bigsqcap \exists s_m \sqcap \bigsqcap \forall s_n : D_n$$

2. For each B_l in D, is there an A_i in C with $A_i = B_l$?
3. For each $\exists s_m$ in D, is there an $\exists r_j$ in C with $s_m = r_j$?
4. For each $\forall s_n : D_n$ in D, is there a $\forall r_k : C_k$ in C such that $C_k \sqsubseteq D_n$ and $s_n = r_k$?

$\rightsquigarrow$ $C \sqsubseteq D$ iff all questions are answered positively

Soundness

Theorem (Soundness)

SUB$(C,D) \Rightarrow C \sqsubseteq D$

Proof sketch.

Reordering of terms *(1)*:

a) Commutativity and associativity are trivial

b) Value restriction law. We show: $\left(\forall r.(C \sqcap D)\right)^{I} = \left(\forall r.C \sqcap \forall r.D\right)^{I}$

Assumption: $d \in \left(\forall r.(C \sqcap D)\right)^{I}$

Case 1: $\nexists e : (d,e) \in r^I$ $\surd$

Case 2: $\exists e : (d,e) \in r^I \Rightarrow e \in (C \sqcap D)^I \Rightarrow e \in C^I,\ e \in D^I$

Since e is arbitrary: $d \in (\forall r.C)^I$, $d \in (\forall r.D)^I$ then d must also be conjunction, i.e., $\left(\forall r.(C \sqcap D)\right)^{I} \subseteq (\forall r.C \sqcap \forall r.D)^I$

Other direction is similar

(2+3+4): Induction on the nesting depth of $\forall$-expressions $\square$

Completeness

Theorem (Completeness)

$C \sqsubseteq D \Rightarrow$ *SUB*(C,D)

Proof idea.

One shows the contrapositive:

$$\neg\text{SUB}(C,D) \Rightarrow C \not\sqsubseteq D$$

Idea: If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element d such that

$$d \in C^I, \text{ but } d \notin D^I$$

$\square$

Generalizing the Algorithm

Extensions of $\mathcal{FL}^-$ by

- $\neg A$ (*atomic negation*),
- $(\leq n\,r)$, $(\geq n\,r)$ (*cardinality restrictions*),
- $r \circ s$ (*role composition*)

does not lead to any problems.
However: If we use full existential restrictions, then it is very unlikely that we can come up with a *simple* structural subsumption algorithm – having the same flavor as the one above.
More precisely: There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison
Reason: Subsumption for $\mathcal{FL}^- + \exists r.C$ is NP-hard (Nutt).

ABox Reasoning

Idea: *abstraction* + *classification*

- *Complete* ABox by propagating value restrictions to role fillers
- Compute for each object its *most specialized concepts*
- These can then be handled using the ordinary subsumption algorithm

Tableau Method

- **Logic $\mathcal{ALC}$**
  - $C \sqcap D$
  - $C \sqcup D$
  - $\neg C$
  - $\forall r.C$
  - $\exists r.C$
- **Idea**: Decide (un-)satisfiability of a concept description C by trying to *systematically construct* a model for C. If that is successful, C is satisfiable. Otherwise C is unsatisfiable.

Example: Subsumption in a TBox

TBox

$\texttt{Hermaphrodite} \doteq \texttt{Male} \sqcap \texttt{Female}$
$\texttt{Parents-of-sons-and-daughters} \doteq$
$\exists\texttt{has-child.Male} \sqcap \exists\texttt{has-child.Female}$
$\texttt{Parents-of-hermaphrodite} \doteq \exists\texttt{has-child.Hermaphrodite}$

Query

$\texttt{Parents-of-sons-and-daughters} \sqsubseteq_{\mathcal{T}}$
$\texttt{Parents-of-hermaphrodites}$

Reductions

1. *Unfolding*
   $\exists\texttt{has-child.Male} \sqcap \exists\texttt{has-child.Female} \sqsubseteq \exists\texttt{has-child.}(\texttt{Male} \sqcap \texttt{Female})$
2. *Reduction to unsatisfiability*
   Is
   $\exists\texttt{has-child.Male} \sqcap \exists\texttt{has-child.Female} \sqcap \neg(\exists\texttt{has-child.}(\texttt{Male} \sqcap \texttt{Female}))$
   unsatisfiable?
3. *Negation normal form* (move negations inside):
   $\exists\texttt{has-child.Male} \sqcap \exists\texttt{has-child.Female} \sqcap \forall\texttt{has-child.}(\neg\texttt{Male} \sqcup \neg\texttt{Female})$
4. *Try to construct a model*

Model Construction (1)

1. **Assumption**: There exists an object x in the interpretation of our concept:

$$x \in (\exists \ldots)^{I}$$

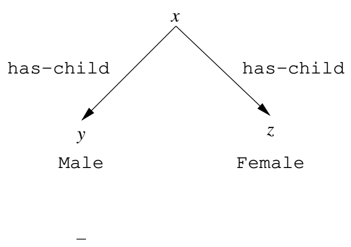
2. This implies that x is in the interpretation of all conjuncts:

$$\begin{array}{lcl} x & \in & (\exists\texttt{has-child.Male})^{I} \\ x & \in & (\exists\texttt{has-child.Female})^{I} \\ x & \in & (\forall\texttt{has-child.}(\neg\texttt{Male} \sqcup \neg\texttt{Female}))^{I} \end{array}$$

3. This implies that there should be objects y and z such that $(x,y) \in \texttt{has-child}^{I}$, $(x,z) \in \texttt{has-child}^{I}$, $y \in \texttt{Male}^{I}$ and $z \in \texttt{Female}^{I}$ and ...

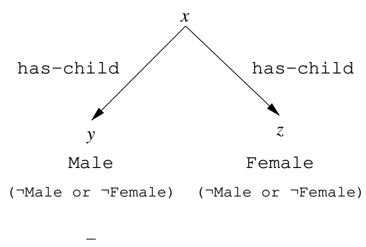
Model Construction (2)

$x : \exists\texttt{has-child.Male}$
$x : \exists\texttt{has-child.Female}$



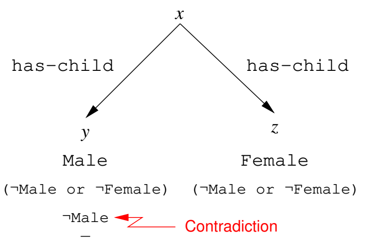
Model Construction (3)

$x : \exists\texttt{has-child.Male}$
$x : \exists\texttt{has-child.Female}$
$x : \forall\texttt{hat-child.}(\neg\texttt{Male} \sqcup \neg\texttt{Female})$



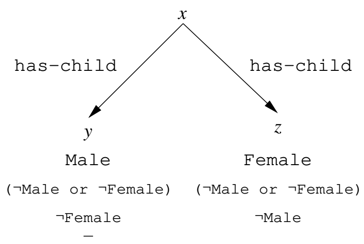
Model Construction (4)

x : ∃`has-child.Male`
x : ∃`has-child.Female`
x : ∀`hat-child.`(¬`Male` ⊔ ¬`Female`)
y : ¬`Male`



Model Construction (5)

x : ∃`has-child.Male`
x : ∃`has-child.Female`
x : ∀`hat-child.`(¬`Male` ⊔ ¬`Female`)
y : ¬`Female`
z : ¬`Male`



$\rightsquigarrow$ Model **constructed**!

Tableau Method (1): NNF

$C \equiv D$ iff $C \sqsubseteq D$ and $D \sqsubseteq C$.
Now we have the following equivalences:

$$\begin{aligned}
\neg(C \sqcap D) &\equiv \neg C \sqcup \neg D \\
\neg(C \sqcup D) &\equiv \neg C \sqcap \neg D \\
\neg\neg C &\equiv C \\
\neg(\forall r.C) &\equiv \exists r.\neg C \\
\neg(\exists r.C) &\equiv \forall r.\neg C
\end{aligned}$$

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: **negation normal form (NNF)**

Theorem (NNF)

The negation normal form of an $\mathcal{ALC}$ concept can be computed in polynomial time.

Tableau Method (2): Constraint Systems

A **constraint** is a syntactical object of the form: $\boxed{x: C}$ or $\boxed{xry}$, where C is a concept description in NNF, r is a role name and x and y are *variable names*.
Let I be an interpretation. An **I-assignment** α is a function that maps each variable symbol to an object of the universe $\mathcal{D}$.
A *constraint* x: C (xry) is **satisfied** by an I-assignment α, if $\alpha(x) \in C^I$ $((\alpha(x), \alpha(y)) \in r^I)$.
A **constraint system** S is a finite, non-empty set of constraints. An I-assignment α satisfies S if α satisfies each constraint in S. S is **satisfiable** if there exists I and α such that α satisfies S.

Theorem

An $\mathcal{ALC}$ concept C in NNF is satisfiable iff the system $\{x: C\}$ is satisfiable.

Tableau Method (3): Transforming Constraint Systems

Transformation rules:

1. $S \rightarrow_{\sqcap} \{x: C_1, x: C_2\} \cup S$
   if $(x: C_1 \sqcap C_2) \in S$ and either $(x: C_1)$ or $(x: C_2)$ or both are not in S.
2. $S \rightarrow_{\sqcup} \{x: D\} \cup S$
   if $(x: C_1 \sqcup C_2) \in S$ and neither $(x: C_1) \in S$ nor $(x: C_2) \in S$ and $D = C_1$ *or* $D = C_2$.
3. $S \rightarrow_{\exists} \{xry, y: C\} \cup S$
   if $(x: \exists r.C) \in S$, y is a *fresh variable*, and there is no z s.t. $(xrz) \in S$ and $(z: C) \in S$.
4. $S \rightarrow_{\forall} \{y: C\} \cup S$
   if $(x: \forall r.C), (xry) \in S$ and $(y: C) \notin S$.

Deterministic rules (1,3,4) vs. non-deterministic (2).
Generating rules (3) vs. non-generating (1,2,4).

Tableau Method (4): Invariances

Theorem (Invariance)

Let S and T be constraint systems:

1. *If T has been generated by applying a deterministic rule to S, then S is satisfiable iff T is satisfiable.*
2. *If T has been generated by applying a non-deterministic rule to S, then S is satisfiable if T is satisfiable. Furthermore, if a non-deterministic rule can be applied to S, then it can be applied such that S is satisfiable iff the resulting system T is satisfiable.*

Theorem (Termination)

Let C be an $\mathcal{ALC}$ concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system $\{x: C\}$.

Tableau Method (5): Soundness and Completeness

A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form $x: A$ and $x: \neg A$, where A is a concept name.

Theorem (Soundness and Completeness)

A closed constraint system is satisfiable iff it does not contain a clash.

Proof idea.

$\Rightarrow$: obvious. $\Leftarrow$: Construct a model by using the concept labels. □

Space Requirements

Because the tableau method is *non-deterministic* ($\rightarrow_{\sqcup}$ rule) ... there could be exponentially many closed constraint systems in the end.
Interestingly, even one constraint system can have *exponential size*.
Example:

$$\begin{array}{l} \exists r.A \sqcap \exists r.B \sqcap \\ \forall r.\Big(\exists r.A \sqcap \exists r.B \sqcap \\ \quad \forall r.(\exists r.A \sqcap \exists r.B \sqcap \\ \qquad \forall r.(\ldots)) \Big) \end{array}$$

However: One can modify the algorithm so that it needs only poly. space.
Idea: Generating a y only for one $\exists r.C$ and then proceeding into the depth.

ABox Reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form $\boxed{x \neq y}$ (for *UNA*):

- *Normalize* and *unfold* and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in $\mathcal{ALC}$ we do not need this because we are never *forced* to identify two objects.

Literature

- Hector J. Levesque and Ronald J. Brachman. Expressiveness and tractability in knowledge representation and reasoning. *Computational Intelligence*, 3:78–93, 1987.
- Manfred Schmidt-Schauß and Gert Smolka. Attributive concept descriptions with complements. *Artificial Intelligence*, 48:1–26, 1991.
- Bernhard Hollunder and Werner Nutt. Subsumption Algorithms for Concept Languages. DFKI Research Report RR-90-04. DFKI, Saarbrücken, 1990. Revised version of paper that was published at ECAI-90.
- F. Baader and U. Sattler. An Overview of Tableau Algorithms for Description Logics. *Studia Logica*, 69:5-40, 2001.
- I. Horrocks, U. Sattler, and S. Tobies. Practical Reasoning for Very Expressive Description Logics. *Logic Journal of the IGPL*, 8(3):239-264, May 2000.