Modal Logics

Knowledge Representation and Reasoning

November 7, 2005

Outline

Motivation for Studying Modal Logics

- Notions like believing and knowing require a more general semantics than e.g. propositional logic has.
- Some KR formalisms can be understood as (fragments of) a propositional modal logic.
- Application 1: spatial representation formalism RCC8.
- Application 2: description logics

Motivation for Modal Logics

Often, we want to state something where we have an "embedded proposition":

- John believes that it is Sunday.
- I know that $2^{10} = 1024$.

Reasoning with embedded propositions:

- John believes that if it is Sunday then shops are closed.
- John believes that it is Sunday.
- This implies (assuming *belief* is closed under *modus ponens*):
  - John believes that shops are closed.

$\rightsquigarrow$ How to formalize this?

Syntax

Propositional logic $+$ operators $\Box$ & $\Diamond$ (*Box & Diamond*):

$$
\begin{array}{lcll}
\varphi & \longrightarrow & \ldots & \textit{classical propositional formula} \\
 & | & \Box\varphi' & \textit{Box} \\
 & | & \Diamond\varphi' & \textit{Diamond}
\end{array}
$$

$\Box$ and $\Diamond$ have the same operator precedence as $\neg$.
Some possible readings of $\Box\varphi$:

- Necessarily φ (alethic)
- Always φ (temporal)
- φ should be true (deontic)
- Agent A believes φ (doxastic)
- Agent A knows φ (epistemic)

$\rightsquigarrow$ different semantics for different intended readings

Truth Functional Semantics?

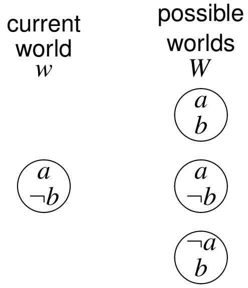
- Could it be possible to define the meaning of $\Box\varphi$ truth functionally, i.e. by referring to the truth value of φ only?
- An attempt to interpret *necessity* truth-functionally:
  - If φ is false, then $\Box\varphi$ should be false.
  - If φ is true, then ...
    - ... $\Box\varphi$ should be true $\rightsquigarrow$ $\Box$ is the identity function
    - ... $\Box\varphi$ should be false $\rightsquigarrow$ $\Box\varphi$ is identical to falsity
- Note: There are only 4 different unary Boolean functions $\{T,F\} \to \{T,F\}$.

Semantics: The Idea

In classical propositional logic, formulae are interpreted over single interpretations and are evaluated to *true* or *false*.
In modal logics one considers sets of interpretations: possible worlds (physically possible, conceivable, ...).
Main idea:

- Consider a world (interpretation) w and a *set of worlds W* which are possible with respect to w.
- A classical formula (with no modal operators) φ is true with respect to (w,W) iff φ is true in w.
- $\Box\varphi$ is true wrt (w,W) iff φ is true in all worlds in W.
- $\Diamond\varphi$ is true wrt (w,W) iff φ is true in one world in W.
- Meanings of $\Box$ and $\Diamond$ are interrelated by $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

Semantics: An Example



Examples:

- $a \wedge \neg b$ is true relative to (w,W).
- $\Box a$ is not true relative to (w,W).
- $\Box(a \vee b)$ is true relative to (w,W).

Question: How to evaluate **modal** formulae in $w \in W$?
$\rightsquigarrow$ For each world, we specify a set of possible worlds.
$\rightsquigarrow$ frames

Frames, Interpretations, and Worlds

A frame is a pair $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set (of *worlds*) and $R \subseteq W \times W$ (the *accessibility relation*).
For $(w, v) \in R$ we write also wRv.
We say that v is an R-successor of w and that v is reachable (or R-reachable) from w.
A (Σ)-interpretation (or model) based on the frame $\mathcal{F} = \langle W, R \rangle$ is a triple $I = \langle W, R, \pi \rangle$, where π is a function from worlds to truth assignments:

$$\pi \colon W \to (\Sigma \to \{T, F\})$$

Semantics: Truth in one World

A formula φ is true in world w of an interpretation $I = \langle W, R, \pi \rangle$ under the following conditions:

$$\begin{array}{rcl}
I, w \models a & \textit{iff} & \pi(w)(a) = T \\
I, w \models \top & & \\
I, w \not\models \bot & & \\
I, w \models \neg\varphi & \textit{iff} & I, w \not\models \varphi \\
I, w \models \varphi \wedge \psi & \textit{iff} & I, w \models \varphi \text{ and } I, w \models \psi \\
I, w \models \varphi \vee \psi & \textit{iff} & I, w \models \varphi \text{ or } I, w \models \psi \\
I, w \models \varphi \to \psi & \textit{iff} & \text{if } I, w \models \varphi \text{ then } I, w \models \psi \\
I, w \models \varphi \leftrightarrow \psi & \textit{iff} & I, w \models \varphi \text{ if and only if } I, w \models \psi \\
I, w \models \Box\varphi & \textit{iff} & I, u \models \varphi \text{ for all } u \text{ s.t. } wRu \\
I, w \models \Diamond\varphi & \textit{iff} & I, u \models \varphi \text{ for at least one } u \text{ s.t. } wRu
\end{array}$$

Satisfiability and Validity

A formula φ is satisfiable in an interpretation I (or *in a frame* $\mathcal{F}$, or *in a class of frames* $\mathcal{C}$) if there exists a world in I (or an interpretation I based on $\mathcal{F}$, or an interpretation I based on a frame contained in the class $\mathcal{C}$, respectively) such that $I, w \models \varphi$.
A formula φ is true in an interpretation I (symbolically $I \models \varphi$) if φ is true in all worlds of I.
A formula φ is valid in a frame $\mathcal{F}$ or $\mathcal{F}$*-valid* (symbolically $\mathcal{F} \models \varphi$) if φ is true in all interpretations based on $\mathcal{F}$.
A formula φ is valid in a class of frames $\mathcal{C}$ or $\mathcal{C}$*-valid* (symbolically $\mathcal{C} \models \varphi$) if $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathcal{C}$.
K is the class of all frames – named after *Saul Kripke*, who invented this semantics.

Validity: Some Examples

1. $\varphi \vee \neg\varphi$
2. $\Box(\varphi \vee \neg\varphi)$
3. $\Box\varphi$, if φ is a classical tautology
4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (*axiom schema K*)

Validity: Some Examples

Theorem

*K is **K**-valid.* $(K = \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi))$

Proof.

Let I be an interpretation and let w be a world in I.
Assumption: $I, w \models \Box(\varphi \to \psi)$, i.e., in all worlds u with wRu, if φ is true then also ψ is.
(Otherwise K is true in any case.)
If $\Box\varphi$ is false in w, then $(\Box\varphi \to \Box\psi)$ is true and K is true in w.
If $\Box\varphi$ is true in w, then both $\Box(\varphi \to \psi)$ and $\Box\varphi$ are true in w. Hence both $\varphi \to \psi$ and φ are true in every world u accessible from w. Hence ψ is true in any such u, and therefore $w \models \Box\psi$. Since I and w were arbitrary, the argument goes through for any I, w, i.e., K is **K**-valid. □

Non-validity: Example

Proposition

*$\Diamond\top$ is not **K**-valid.*

Proof.

A counterexample is the following interpretation:

$$I = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto T)\} \rangle.$$

We have $I, w \not\models \Diamond\top$ because there is no u such that wRu. □

Non-validity: Example

Proposition

*$\Box\varphi \to \varphi$ is not **K**-valid.*

Proof.

A counterexample is the following interpretation:

$$I = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto F)\} \rangle.$$

We have $I, w \models \Box a$ but $I, w \not\models a$. □

Non-validity: Another Example

Proposition

*$\Box\varphi \to \Box\Box\varphi$ is not **K**-valid.*

Proof.

A counterexample is the following interpretation:

$$I = \langle \{u, v, w\}, \{(u, v), (v, w)\}, \pi \rangle$$

with

$$\begin{aligned} \pi(u) &= \{a \mapsto T\} \\ \pi(v) &= \{a \mapsto T\} \\ \pi(w) &= \{a \mapsto F\} \end{aligned}$$

This means $I, u \models \Box a$, but $I, u \not\models \Box\Box a$. □

Accessibility and Axiom Schemata

Let us consider the following axiom schemata:

T: $\Box\varphi \rightarrow \varphi$ (*knowledge axiom*)
4: $\Box\varphi \rightarrow \Box\Box\varphi$ (*positive introspection*)
5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ (or $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: *negative introspection*)
B: $\varphi \rightarrow \Box\Diamond\varphi$
D: $\Box\varphi \rightarrow \Diamond\varphi$ (or $\Box\varphi \rightarrow \neg\Box\neg\varphi$: *disbelief in the negation*)

. . . and the following classes of frames, for which the accessibility relation is restricted as follows:

T: reflexive (wRw for each world w)
4: transitive (wRu and uRv implies wRv)
5: euclidian (wRu and wRv implies uRv)
B: symmetric (wRu implies uRw)
D: serial (for each w there exists v with wRv)

Connection between Accessibility Relations and Axiom Schemata (1)

Theorem
Axiom schema T $(4,5,B,D)$ is **T***- valid (***4-, 5-, B-***, or* **D***-valid, respectively).*

Proof.
For T and **T**:Let $\mathcal{F}$ be a frame from class **T**. Let I be an interpretation based on $\mathcal{F}$ and let w be an arbitrary world in I. If $\Box\varphi$ is not true in a world w, then axiom T is true in w. If $\Box\varphi$ is true in w, then φ is true in all accessible worlds.Since the accessibility relation is reflexive, w is among the accessible worlds, i.e., φ is true in w. This means that also in this case T is true w. This means, T is true in all worlds in all interpretations based on **T**-frames. □

Connection between Accessibility Relations and Axiom Schemata (2)

Theorem
If T $(4,5,B,D)$ is valid in a frame $\mathcal{F}$, then $\mathcal{F}$ is a **T***-Frame (***4-, 5-, B-***, or* **D***-frame, respectively).*

Proof.
For T and **T**: Assume that $\mathcal{F}$ is not a **T**-frame. We will construct an interpretation based on $\mathcal{F}$ that falsifies T.
Because $\mathcal{F}$ is not a **T**-frame, there is a world w such that not wRw.
Construct an interpretation I such that $w \not\models p$ and $v \models p$ for all v such that wRv.
Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$. □

Different Modal Logics

Name	Property	Axiom schema
K	–	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
T	reflexivity	$\Box\varphi \rightarrow \varphi$
4	transitivity	$\Box\varphi \rightarrow \Box\Box\varphi$
5	euclidicity	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$
B	symmetry	$\varphi \rightarrow \Box\Diamond\varphi$
D	seriality	$\Box\varphi \rightarrow \Diamond\varphi$

Some basic modal logics:

$$\begin{array}{rcl} K & & \\ KT4 & = & S4 \\ KT5 & = & S5 \\ \vdots & & \end{array}$$

Different Modal Logics

logics	$\Box$	$\Diamond = \neg\Box\neg$	K	T	4	5	B	D
alethic	necessarily	possibly	Y	Y	Y	Y	Y	Y
epistemic	known	possible	Y	Y	Y	Y	Y	Y
doxastic	believed	possible	Y	N	Y	Y	N	Y
deontic	obligatory	permitted	Y	N	?/Y	?/Y	N	Y
temporal	always in the future	sometimes	Y	Y/N	Y	N	N	N/Y

Proof Methods

- How can we show that a formula is $\mathcal{C}$-valid?
- In order to show that a formula is not $\mathcal{C}$-valid, one can construct a counterexample (= an interpretation that falsifies it.)
- When trying out all ways of generating a counterexample without success, this counts as a proof of validity.
- method of (analytic/semantic) tableaux

Tableau Method

A tableau is a tree with nodes marked as follows:

- $w \models \varphi$,
- $w \not\models \varphi$, and
- wRv.

A branch that contains nodes marked with $w \models \varphi$ and $w \not\models \varphi$ is closed. All other branches are open. If all branches are closed, the tableau is closed.
A tableau is constructed by using the tableau rules.

Tableau Rules for the Propositional Logic

$$\frac{w \models \varphi \lor \psi}{w \models \varphi \mid w \models \psi} \qquad \frac{w \not\models \varphi \lor \psi}{\begin{array}{c} w \not\models \varphi \\ w \not\models \psi \end{array}} \qquad \frac{w \models \neg\varphi}{w \not\models \varphi}$$

$$\frac{w \models \varphi \land \psi}{\begin{array}{c} w \models \varphi \\ w \models \psi \end{array}} \qquad \frac{w \not\models \varphi \land \psi}{w \not\models \varphi \mid w \not\models \psi} \qquad \frac{w \not\models \neg\varphi}{w \models \varphi}$$

$$\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi \mid w \models \psi} \qquad \frac{w \not\models \varphi \rightarrow \psi}{\begin{array}{c} w \models \varphi \\ w \not\models \psi \end{array}}$$

Additional Tableau Rules for the Modal Logic K

$$\frac{w \models \Box\varphi}{v \models \varphi}$$ if wRv is on the branch already

$$\frac{w \not\models \Box\varphi}{\begin{array}{c} wRv \\ v \not\models \varphi \end{array}}$$ for new v

$$\frac{w \models \Diamond\varphi}{\begin{array}{c} wRv \\ v \models \varphi \end{array}}$$ for new v

$$\frac{w \not\models \Diamond\varphi}{v \not\models \varphi}$$ if wRv is on the branch already

Properties of K Tableaux

Proposition

If a K-tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness)

If a K-tableau with root $w \not\models \varphi$ is closed, then φ is ***K****-valid.*

Theorem (Completeness)

If φ is ***K****-valid, then there is a closed tableau with root $w \not\models \varphi$.*

Proposition (Termination)

There are strategies for constructing ***K****-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.*

Tableau Rules for Other Modal Logics

Proofs within more restricted classes of frames allow the use of further tableau rules.

- For reflexive (**T**) frames we may extend any branch with wRw.
- For transitive (**4**) frames we have the following additional rule:
  - If wRv and vRu are in a branch, wRu may be added to the branch.
- For serial (**D**) frames we have the following rule:
  - If there is $w \models \ldots$ or $w \not\models \ldots$ on a branch, then add wRv for a new world v.
- Similar rules for other properties...

Testing Logical Consequence with Tableaux

- Let Θ be a set of formulas. When does a formula φ follow from Θ: $\Theta \models_{\mathbf{X}} \varphi$?
- Test whether in all interpretations on $\mathbf{X}$-frames in which Θ is true, also φ is true.
- Wouldn't there be a deduction theorem we could use?
- Example: $a \models_{\mathbf{K}} \Box a$ holds, but $a \rightarrow \Box a$ is not **K**-valid.
- There is no deduction theorem as in the propositional logic, and logical consequence cannot be directly reduced to validity!

Tableaus and Logical Implication

For testing logical consequence, we can use the following tableau rule:

- If w is a world on a branch and $\psi \in \Theta$, then we can add $w \models \psi$ to our branch.
- Soundness is obvious.
- Completeness is non-trivial.

Connection between propositional modal logic and FOL?

- There are similarities between the predicate logic and propositional modal logics:
  1. $\Box$ vs. $\forall$
  2. $\Diamond$ vs. $\exists$
  3. the possible worlds vs. the objects of the universe
- In fact, we can show for many propositional modal logics that they can be embedded in the predicate logic. $\Longrightarrow$ Modal logics can be understood as a sublanguage of FOL.

Embedding Modal Logics in the Predicate Logic (1)

1. $\tau(p,x) = p(x)$ for propositional variables p
2. $\tau(\neg\phi,x) = \neg\tau(\phi,x)$
3. $\tau(\phi \vee \psi,x) = \tau(\phi,x) \vee \tau(\psi,x)$
4. $\tau(\phi \wedge \psi,x) = \tau(\phi,x) \wedge \tau(\psi,x)$
5. $\tau(\Box\phi,x) = \forall y(R(x,y) \rightarrow \tau(\phi,y))$ for some new y
6. $\tau(\Diamond\phi,x) = \exists y(R(x,y) \wedge \tau(\phi,y))$ for some new y

Embedding Modal Logics in the Predicate Logic (2)

Theorem

ϕ is K-valid if and only if $\forall x\tau(\phi,x)$ is valid in the predicate logic.

Theorem

ϕ is T-valid if and only if in the predicate logic the logical consequence $\{\forall xR(x,x)\} \models \forall x\tau(\phi,x)$ holds.

Example

$((\Box p) \wedge \Diamond(p \rightarrow q)) \rightarrow \Diamond q$ is K-valid because

$$\begin{aligned} \forall x((\forall x'(R(x,x') \rightarrow p(x'))) \wedge \exists x'(R(x,x') \wedge (p(x') \rightarrow q(x')))) \\ \rightarrow \exists x'(R(x,x') \wedge q(x')) \end{aligned}$$

is valid in the predicate logic.

Outlook

We only looked at some basic propositional modal logics. There are also

- modal first order logics (with quantification $\forall$ and $\exists$ and predicates)
- multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents
- temporal and dynamic logics (modalities that refer to time or programs, respectively)

Outlook

Did we really do something new? Couldn't we have done everything in propositional modal logic in FOL already?

- Yes – but now we know much more about the (restricted) system and have decidable problems!

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