Complexity Theory

Knowledge Representation and Reasoning

November 2, 2005

Outline

Motivation

Reminder: Basic Notions

- Algorithms and Turing Machines
- Problems, Solutions, and Complexity
- Complexity Classes P and NP
- Upper and Lower Bounds
- Polynomial Reductions
- NP-Completeness

Beyond NP

- The Class co-NP
- The Class PSPACE
- Other Classes

Oracle TMs and the Polynomial Hierarchy

- Oracle Turing-Machines
- Turing Reduction
- Complexity Classes Based on OTMs
- QBF

Motivation for Using Complexity Theory

- Complexity theory can answer us questions how easy or hard a problem is

⇝ Gives hints on what appropriate algorithms could be,e.g.,
  - algorithms for polynomial-time problems are usually easy to design
  - for NP-complete problems, backtracking and local search work well

⇝ Gives us hint on what type of algorithms will (most probably) not work
  - for problem that are believed to be harder than NP-complete ones, simple backtracking will not work

⇝ Gives hint on what sub-problem might be interesting

Algorithms and Turing Machines

- We use Turing machines as formal models of algorithms
- This is justified, because
  - we assume that Turing machines can compute all computable functions
  - the resource requirements (in term of time and memory) of a Turing machine are only polynomially worse than other models
- The regular type of Turing machine is the deterministic one: DTM or simply TM
- Often, however, we use the notion of nondeterministic TMs: NDTM

Problems, Solutions, and Complexity

- A **problem** is a set of pairs (I, A) of strings in $\{0, 1\}^*$.
  I: Instance
  A: Answer. If $A \in \{0, 1\}$: *decision problem*

$\rightsquigarrow$ A decision problem is the same as a *formal language* (namely the set of strings formed by the instances with answer 1)

- An algorithm *decides* (or solves) a problem if it computes the right answer for all instances.
- The **complexity of an algorithm** is a function

$$T\colon \mathbf{N} \rightarrow \mathbf{N},$$

  measuring the *number of basic steps* (or memory requirement) the algorithm needs to compute an answer depending on the *size* of the instance.
- The **complexity of a problem** is the complexity of the most efficient algorithm that solves this problem.

Complexity Classes P and NP

Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: P

$\longrightarrow$ Problems in **P** are said to be efficiently solvable (although this might not be true if the exponent is very large)

$\leadsto$ In practice, this notion appears to be more *often reasonable* than not

- The class of problems decidable on **non**deterministic Turing machines in polynomial time: NP
- More classes are definable using other resource bounds on time and memory.

Upper and Lower Bounds

- **Upper bounds** (*membership* in a class) are usually easy to prove:
  - ⇝ provide an algorithm
  - ⇝ show that the resource bounds are respected.
- **Lower bounds** (*hardness* for a class) are usually difficult to show.
  - ⇝ the technical tool here is the polynomial reduction (or any other appropriate reduction).
  - ⇝ show that some hard problem can be reduced to the problem at hand

Polynomial Reductions

- Given two languages L_1 and L_2, L_1 can be *polynomially reduced to* L_2, written $L_1 \leq_p L_2$, iff there exists a polynomially computable function f such that

$$x \in L_1 \text{ iff } f(x) \in L_2$$

$\longrightarrow$ It cannot be harder to decide L_1 than L_2

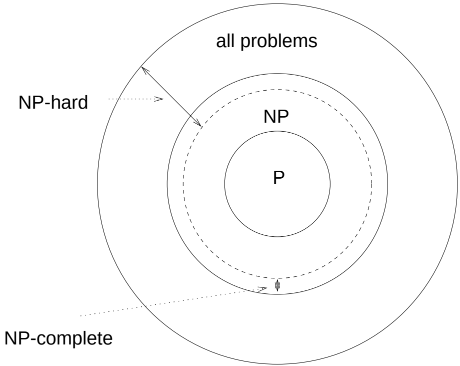
$\longrightarrow$ L is hard for a class C (C-hard) iff all languages of this class reduce to it.

$\longrightarrow$ L is complete for C (C-complete) iff it is hard and $L \in C$.

NP-complete Problems

- A problem is **NP-complete** iff it is **NP-hard** and in NP.
- Example: **SAT** – the satisfiability problem for propositional logic – is NP-complete (Cook/Karp)

→ Membership is obvious, hardness follows because computations on a NDTM correspond to satisfying truth-assignments of certain formulae



The Complexity Class co-NP

- Note that there is some *asymmetry* in the definition of NP.
  - It is clear that we can decide SAT by using a NDTM with polynomially bounded computation
  - ⇝ There exists an accepting computation of polynomial length iff the formula is satisfiable
  - What if we want to solve UNSAT, the complementary problem?
  - ⇝ It seems necessary to check *all* possible truth-assignments!
- Define co-$C = \{L | \Sigma^* - L \in C\}$, provided Σ is our alphabet
- ⇝ co-NP = $\{L | \Sigma^* - L \in \text{NP}\}$
- For example UNSAT, TAUT $\in$ co-NP!
- Note: P is closed under complement, i.e.,

$$\text{P} \in \text{NP} \cap \text{co-NP}$$

PSPACE

There are problems even more difficult than NP and co-NP.

Definition ((N)PSPACE)

PSPACE (NPSPACE) is the class of decision problems that can be decided on **deterministic** (**non-deterministic**) Turing machines using only polynomial many tape cells.

Some facts about PSPACE:

- PSPACE is closed under complements (as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- NP$\subseteq$PSPACE (because in polynomial time one can "visit" only polynomial space, i.e., NP$\subseteq$NPSPACE)
- It is unknown whether NP$\neq$PSPACE, but it is **believed** that this is true.

PSPACE-completeness

Definition (PSPACE-completeness)

A decision problem (or language) is **PSPACE-complete**, if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, PSPACE-complete problems are the “hardest” problems in PSPACE (similar to NP-completeness). They appear to be “harder” than **NP-complete** problems from a *practical point of view*.

An example for a PSPACE-complete problem is the NDFA equivalence problem:

Instance: Two non-deterministic finite state automata A_1 and A_2.

Question: Are the languages accepted by A_1 and A_2 identical?

Other Complexity Classes . . .

- There are complexity classes above PSPACE (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME, . . .),
- there are (infinitely many) classes between NP and PSPACE (the polynomial hierarchy defined by oracle machines)
- there are (infinitely many) classes inside P (circuit classes with different depths)

→ and for most of the classes *we do not know* whether the containment relationships are strict

Oracle Turing Machines

- An Oracle Turing machine ((N)OTM) is a Turing machine (DTM, NDTM) with the possibility to query an oracle, another Turing machine without resource restrictions, whether it accepts or reject a given string.

$\longrightarrow$ Computation by the oracle does not cost anything!

- Formalization:
  - a tape onto which strings for the oracle are written,
  - a yes/no answer from the oracle depending on whether it accepts or rejects the input string.

$\rightsquigarrow$ Usage of OTMs answers what-if questions: What if we could solve the oracle-problem efficiently?

Turing Reductions

- OTMs allow us to define a more *general type of reduction*
- *Idea*: The “classical” reduction can be seen as calling a subroutine once.
- L_1 is Turing-reduced to L_2, symbolically $L_1 \leq_T L_2$ if there exists an OTM that decides L_1 by using an oracle for L_2.
- Polynomial reducibility implies Turing reducibility, but not *vice versa*!

$\rightsquigarrow$ NP-completeness and co-NP-completeness with respect to Turing reducibility are identical!

$\longrightarrow$ Turing reducibility can also be applied to general search problems!

Complexity Classes Based on Oracle TMs

1. P^{NP} = decision problems solved by poly-time DTMs with an oracle for a decision problem in NP.
2. NP^{NP} = decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
3. $co\text{-}NP^{NP}$ = complements of decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
4. $NP^{NP^{NP}}$ = ...
   and so on

Example

- Consider the Minimum Equivalent Expression (MEE) problem:

  Instance: A well-formed Boolean formula ϕ using the standard connectives (not $\leftrightarrow$) and a nonnegative integer K.
  Question: Is there a well-formed Boolean formula ϕ' that contains K or fewer literal occurrences and that is logical equivalent to ϕ?

- This problem is NP-hard (writ. to Turing reductions).
- It does not appear to be NP-complete
- We could guess a formula and then use a SAT-oracle

$\rightsquigarrow$ MME $\in$ $\mathrm{NP}^{\mathrm{NP}}$.

The Polynomial Hierarchy

The complexity classes based on OTMs form an infinite hierarchy.

The polynomial hierarchy PH

$$\begin{array}{rclrclrcl} \Sigma_0^p & = & P & \Pi_0^p & = & P & \Delta_0^p & = & P \\ \Sigma_{i+1}^p & = & \mathrm{NP}^{\Sigma_i^p} & \Pi_{i+1}^p & = & \text{co-}\Sigma_{i+1}^p & \Delta_{i+1}^p & = & P^{\Sigma_i^p} \end{array}$$

- PH = $\bigcup_{i\geq 0}(\Sigma_i^p \cup \Pi_i^p \cup \Delta_i^p)$ $\subseteq$PSPACE
- NP = Σ_1^p, co-NP = Π_1^p

Quantified Boolean Formulae: Definition

- If ϕ is a propositional formula, P is the set of Boolean variables used in ϕ and σ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma\phi$ is a QBF.
- A formula $\exists x \phi$ is true if and only if $\phi[\top/x] \vee \phi[\bot/x]$ is true. (Equivalently, $\phi[\top/x]$ *is true or* $\phi[\bot/x]$ *is true*.)
- A formula $\forall x \phi$ is true if and only if $\phi[\top/x] \wedge \phi[\bot/x]$ is true. (Equivalently, $\phi[\top/x]$ *is true and* $\phi[\bot/x]$ *is true*.)
- This definition directly leads to an AND/OR tree traversal algorithm for evaluating QBF.

Quantified Boolean Formulae: Definition

The evaluation problem of QBF generalizes both the *satisfiability* and *validity/tautology problems* of propositional logic.
The latter are respectively NP-complete and co-NP-complete whereas the former is PSPACE-complete.

Example

The formulae $\forall x \exists y (x \leftrightarrow y)$ and $\exists x \exists y (x \wedge y)$ are true.

Example

The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \vee y)$ are false.

The Polynomial Hierarchy: Connection to QBF

Truth of QBFs with prefix $\overbrace{\forall\exists\forall\cdots}^{i}$ is Π_i^p-complete.

Truth of QBFs with prefix $\overbrace{\exists\forall\exists\cdots}^{i}$ is Σ_i^p-complete.

Special cases corresponding to SAT and TAUT:
The truth of QBFs with prefix $\exists x_1^1 \cdots x_n^1$ is NP$= \Sigma_1^p$-complete.
The truth of QBFs with prefix $\forall x_1^1 \cdots x_n^1$ is co-NP$= \Pi_1^p$-complete.

Literature

M. R. Garey and D. S. Johnson.
Computers and Intractability – A Guide to the Theory of NP-Completeness.
Freeman and Company, San Francisco, 1979.

C. H. Papadimitriou.
Computational Complexity.
Addison-Wesley,Reading, MA, 1994.