Classical Logic

Predicate Logic

Knowledge Representation and Reasoning

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Outline

Why First-Order Logic (FOL)?

- In propositional logic, the only building blocks are atomic propositions.
- We cannot talk about the internal structures of these propositions.
- Example:
  - *All CS students know formal logic*
  - *Peter is a CS student*
  - Therefore, *Peter knows formal logic*
  - Not possible in propositional logic
- Idea: We introduce predicates, functions, object variables and quantifiers.

Syntax

- variable symbols: $x, y, z, \ldots$
- n-ary function symbols: $f, g, \ldots$
- constant symbols: $a, b, c, \ldots$
- n-ary predicate symbols: $P, Q, \ldots$
- logical symbols: $\forall$, $\exists$, $=$, $\neg$, $\wedge$, $\ldots$

Terms	t	$\longrightarrow$	x	variable
		$\mid$	$f(t_1, \ldots, t_n)$	function application
		$\mid$	a	constant
Formulae	φ	$\longrightarrow$	$P(t_1, \ldots, t_n)$	atomic formula
		$\mid$	$t = t'$	identity formulae
		$\mid$	$\ldots$	propositional connectives
		$\mid$	$\forall x(\varphi')$	universal quantification
		$\mid$	$\exists x(\varphi')$	existential quantification

ground term, etc.: term, etc. without variable occurrences

Semantics: Idea

- In FOL, the universe of discourse consists of objects, functions over these objects, and relations over these objects.
- Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- Notation: Instead of $I(x)$ we write x^I.
- Note: Usually one considers all possible non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- Satisfiability and validity is then considered wrt all these universes.

Formal Semantics: Interpretations

Interpretations: $I = \langle \mathcal{D}, \cdot^I \rangle$ with $\mathcal{D}$ being an arbitrary non-empty set and I being a function which maps

- n-ary function symbols f to n-ary functions $f^I \in [\mathcal{D}^n \to \mathcal{D}]$,
- constant symbols a to objects $a^I \in \mathcal{D}$, and
- n-ary predicates P to n-ary relations $P^I \subseteq \mathcal{D}^n$.

Interpretation of ground terms:

$$(f(t_1, \ldots, t_n))^I \quad = \quad f^I(t_1{}^I, \ldots, t_n{}^I) \ (\in \mathcal{D})$$

Truth of ground atoms:

$$I \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle t_1{}^I, \ldots, t_n{}^I \rangle \in P^I$$

Examples

$$
\begin{aligned}
\mathcal{D} &= \{d_1, \ldots, d_n\}, n \geq 2 \\
\mathrm{a}^I &= d_1 \\
\mathrm{b}^I &= d_2 \\
\mathrm{eye}^I &= \{d_1\} \\
\mathrm{red}^I &= \mathcal{D} \\
I &\models \mathrm{red(b)} \\
I &\not\models \mathrm{eye(b)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D} &= \{1, 2, 3, \ldots\} \\
1^I &= 1 \\
2^I &= 2 \\
&\vdots \\
\mathrm{even}^I &= \{2, 4, 6, \ldots\} \\
\mathrm{succ}^I &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\
I &\not\models \mathrm{even}(3) \\
I &\models \mathrm{even}(\mathrm{succ}(3))
\end{aligned}
$$

Formal Semantics: Variable Maps

V is the set of variables. Function $\alpha\colon V \longrightarrow \mathcal{D}$ is a variable map.
Notation: $\alpha[x/d]$ is identical to α except for x where $\alpha[x/d](x) = d$.
Interpretation of terms under $\mathcal{I}, \alpha$:

$$
\begin{aligned}
x^{\mathcal{I},\alpha} &= \alpha(x) \\
a^{\mathcal{I},\alpha} &= a^{\mathcal{I}} \\
(f(t_1,\ldots,t_n))^{\mathcal{I},\alpha} &= f^{\mathcal{I}}(t_1{}^{\mathcal{I},\alpha},\ldots,t_n{}^{\mathcal{I},\alpha})
\end{aligned}
$$

Truth of atomic formulae:

$$\mathcal{I},\alpha \models P(t_1,\ldots,t_n) \quad \text{iff} \quad \langle t_1{}^{\mathcal{I},\alpha},\ldots,t_n{}^{\mathcal{I},\alpha}\rangle \in P^{\mathcal{I}}$$

Example (cont.):

$$\alpha = \{x \mapsto d_1, y \mapsto d_2\} \qquad \mathcal{I},\alpha \models \mathrm{red}(x) \qquad \mathcal{I},\alpha[y/d_1] \models \mathrm{eye}(y)$$

Formal Semantics: Truth

Truth of φ by $\mathcal{I}$ under α ($\mathcal{I},\alpha \models \varphi$) is defined as follows.

$$
\begin{array}{rcl}
\mathcal{I},\alpha \models P(t_1,\ldots,t_n) & \text{iff} & \langle t_1{}^{\mathcal{I},\alpha},\ldots,t_n{}^{\mathcal{I},\alpha}\rangle \in P^{\mathcal{I}} \\
\mathcal{I},\alpha \models t_1 = t_2 & \text{iff} & t_1{}^{\mathcal{I},\alpha} = t_2{}^{\mathcal{I},\alpha} \\
\mathcal{I},\alpha \models \neg\varphi & \text{iff} & \mathcal{I},\alpha \not\models \varphi \\
\mathcal{I},\alpha \models \varphi \wedge \psi & \text{iff} & \mathcal{I},\alpha \models \varphi \text{ and } \mathcal{I},\alpha \models \psi \\
\mathcal{I},\alpha \models \varphi \vee \psi & \text{iff} & \mathcal{I},\alpha \models \varphi \text{ or } \mathcal{I},\alpha \models \psi \\
\mathcal{I},\alpha \models \varphi \rightarrow \psi & \text{iff} & \text{if } \mathcal{I},\alpha \models \varphi, \text{then } \mathcal{I},\alpha \models \psi \\
\mathcal{I},\alpha \models \varphi \leftrightarrow \psi & \text{iff} & \mathcal{I},\alpha \models \varphi, \text{ iff } \mathcal{I},\alpha \models \psi \\
\mathcal{I},\alpha \models \forall x\,\varphi & \text{iff} & \mathcal{I},\alpha[x/d] \models \varphi \text{ for all } d \in \mathcal{D} \\
\mathcal{I},\alpha \models \exists x\,\varphi & \text{iff} & \mathcal{I},\alpha[x/d] \models \varphi \text{ for some } d \in \mathcal{D}
\end{array}
$$

Examples

$$\Theta = \left\{ \begin{array}{l} \text{eye}(a), \text{eye}(b) \\ \forall x(\text{eye}(x) \rightarrow \text{red}(x)) \end{array} \right\}$$

$$\mathcal{D} = \{d_1, \ldots, d_n, \}\ n > 1$$

$$\text{a}^I = d_1$$

$$\text{b}^I = d_1$$

$$\text{eye}^I = \{d_1\}$$

$$\text{red}^I = \mathcal{D}$$

$$\alpha = \{(x \mapsto d_1), (y \mapsto d_2)\}$$

Questions:

- $I, \alpha \models \text{eye}(b) \vee \neg\text{eye}(b)$? *Yes*
- $I, \alpha \models \text{eye}(x) \rightarrow \text{eye}(x) \vee \text{eye}(y)$? *Yes*
- $I, \alpha \models \text{eye}(x) \rightarrow \text{eye}(y)$? *No*
- $I, \alpha \models \text{eye}(a) \wedge \text{eye}(b)$? *Yes*
- $I, \alpha \models \forall x(\text{eye}(x) \rightarrow \text{red}(x))$? *Yes*
- $I, \alpha \models \Theta$? *Yes*

Terminology

$\mathcal{I}, \alpha$ is a model of φ iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be satisfiable, unsatisfiable, falsifiable, valid.
Two formulae φ and ψ are logically equivalent $(\varphi \equiv \psi)$ iff for all $\mathcal{I}, \alpha$:

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

Note: $P(x) \not\equiv P(y)$!
Logical Implication is also similar to propositional logic:

$$\Theta \models \varphi \quad \text{iff} \quad \text{for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

Free and Bound Variables

Variables can be free or bound (by a quantifier) in a formula:

$$
\begin{array}{rcl}
\mathsf{free}(x) & = & \{x\} \\
\mathsf{free}(f(t_1,\ldots,t_n)) & = & \mathsf{free}(t_1)\cup\ldots\cup\mathsf{free}(t_n) \\
\mathsf{free}(t_1 = t_2) & = & \mathsf{free}(t_1)\cup\mathsf{free}(t_2) \\
\mathsf{free}(P(t_1,\ldots,t_n)) & = & \mathsf{free}(t_1)\cup\ldots\cup\mathsf{free}(t_n) \\
\mathsf{free}(\neg\varphi) & = & \mathsf{free}(\varphi) \\
\mathsf{free}(\varphi * \psi) & = & \mathsf{free}(\varphi)\cup\mathsf{free}(\psi) \quad * = \vee,\wedge,\rightarrow,\leftrightarrow \\
\mathsf{free}(\Xi x\varphi) & = & \mathsf{free}(\varphi) - \{x\} \quad \Xi = \forall,\exists
\end{array}
$$

Example: $\forall x\ (R(\boxed{y},\boxed{z}) \wedge \exists\ y\ (\neg P(y,x) \vee R(y,\boxed{z})))$
Framed occurrences are free, all others are bound.

Open & Closed Formulae

- Formulae without free variables are called closed formulae or sentences. Formulae with free variables are called open formulae.
- Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of $\forall$ and $\exists$).
- Note that *logical equivalence*, *satisfiability*, and entailment are independent from variable maps if we consider only closed formulae.
- For closed formulae, we omit α in connection with $\models$:

$$I \models \varphi.$$

Important Theorems

Theorem (Compactness)

Let $\Phi \cup \{\psi\}$ be a set of closed formulae.

(a) $\Phi \models \psi$ *iff there exists a finite subset* $\Phi' \subset \Phi$ *s. t.* $\Phi' \models \psi$*.*

(b) Φ *is satisfiable iff each finite subset* $\Phi' \subset \Phi$ *is satisfiable.*

Theorem (Löwenheim-Skolem)

Each countable set of closed formuale that is satisfiable is satisfiable on a countable domain.

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