An Introduction to Game Theory Part V: Extensive Games with Perfect Information

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Motivation

- So far, all games consisted of just one simultaneous move by all players
- Often, there is a whole sequence of moves and player can react to the moves of the other players
- Examples:
  - board games
  - card games
  - negotiations
  - interaction in a market

Example: Entry Game

- An *incumbent* faces the possibility of *entry* by a *challenger*. The *challenger* may enter (*in*) or not enter (out). If it enters, the *incumbent* may either *give in* or *fight.*
- The payoffs are
  - challenger: 1, incumbent: 2 if challenger does not enter
  - challenger: 2, incumbent: 1 if challenger enters and incumbent gives in
  - challenger: 0, incumbent: 0 if challenger enters and incumbent fights

(similar to chicken – but here we have a sequence of moves!)

Formalization: Histories

- The possible developments of a game can be described by a *game tree* or a mechanism to construct a game tree
- Equivalently, we can use the set of paths starting at the root: all potential *histories* of moves
  - potentially infinitely many (infinite branching)
  - potentially infinitely long

Extensive Games with Perfect Information

An ***extensive games with perfect information*** consists of

- a non-empty, finite set of players $N = \{1, \ldots, n\}$
- a set H (histories) of sequences such that
  - $\langle\rangle \in H$
  - H is prefix-closed
  - if for an infinite sequence $\langle a_i \rangle_{i \in \mathbf{N}}$ every prefix of this sequence is in H, then the infinite sequence is also in H
  - sequences that are not a *proper prefix* of another strategy are called terminal histories and are denoted by Z. The elements in the sequences are called actions.
- a player function $P: H \setminus Z \rightarrow N$,
- for each player i a payoff function $u_i: Z \rightarrow \mathbf{R}$

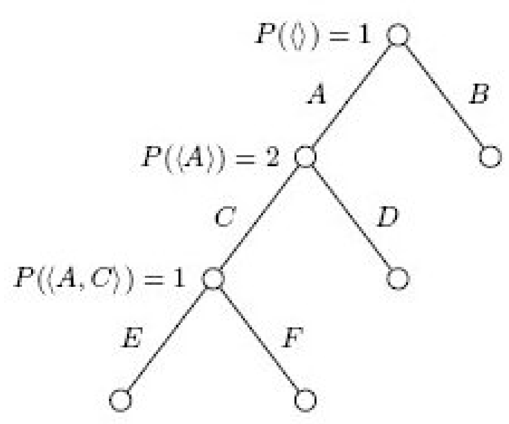
- A game is finite if H is finite
- A game as a finite horizon, if there exists a finite upper bound for the length of histories

Entry Game – Formally

- players N = {1,2} (1: challenger, 2: incumbent)
- histories H = {⟨⟩, ⟨out⟩, ⟨in⟩, ⟨in, fight⟩, ⟨in, give_in⟩}
- terminal histories: Z = {⟨out⟩, in, fight⟩, ⟨in, give_in⟩}
- player function:
  - P(⟨⟩) = 1
  - P(⟨in⟩) = 2
- payoff function
  - u_1(⟨out⟩)=1, u_2(⟨out⟩)=2
  - u_1(⟨in, fight⟩)=0, u_2(⟨in, fight⟩)=0
  - u_1(⟨in,give_in⟩)=2, u_2(⟨in,give_in⟩)=1

Strategies

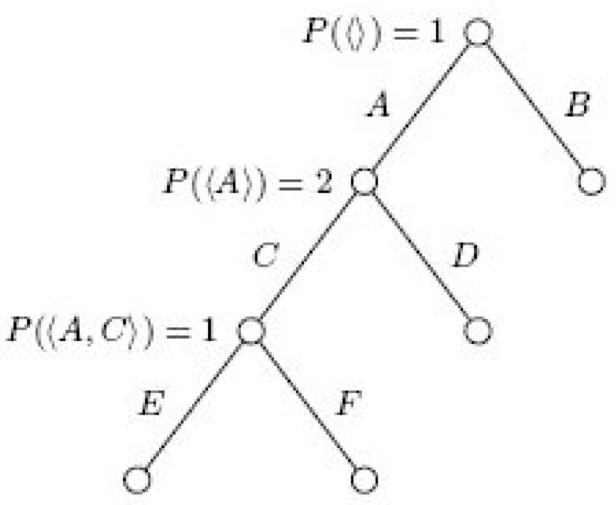
- The number of possible actions after history h is denoted by *A(h)*.
- A strategy for player *i* is a function s_i that maps each history *h* with *P(h)* = *i* to an element of *A(h)*.
- *Notation*: Write strategy as a sequence of actions as they are to be chosen at each point when visiting the nodes in the game tree in breadth-first manner.



- Possible strategies for player 1:
  - AE, AF, BE, BF
- for player 2:
  - C,D
- Note: Also decisions for histories that cannot happen given earlier decisions!

Outcomes

- The outcome *O(s)* of a strategy profile *s* is the terminal history that results from applying the strategies successively to the histories starting with the empty one.
- What is the outcome for the following strategy profiles?
- O(AF,C) =
- O(AF,D) =
- O(BF,C) =



Nash Equilibria in Extensive Games with Perfect Information

- A strategy profile s^* is a Nash Equilibrium in an extensive game with perfect information if for all players i and all strategies s_i of player i:

$$u_i(O(s^*_{-i}, s^*_i)) \geq u_i(O(s^*_{-i}, s_i))$$

- Equivalently, we could define the strategic form of an extensive game and then use the existing notion of Nash equilibrium for strategic games.

The Entry Game - again

- Nash equilibra?
  - In, Give in
  - Out, Fight
- But why should the challenger take the “threat” seriously that the incumbent starts a fight?
- Once the challenger has played “in”, there is no point for the incumbent to reply with “fight”. So “fight” can be regarded as an empty threat

	Give in	Fight
In	2,1	0,0
Out	1,2	1,2

- Apparently, the Nash equilibrium *out, fight* is not a real *“steady state”* – we have ignored the *sequential nature* of the game

Sub-games

- Let $G=(N,H,P,(u_i))$ be an extensive game with perfect information. For any non-terminal history h, the sub-game $G(h)$ following history h is the following game: $G'=(N,H',P',(u_i'))$ such that:
  - H' is the set of histories such that for all h': $(h,h') \in H$
  - $P'(h') = P((h,h'))$
  - $u_i'(h') = u_i((h,h'))$

How many sub-games are there?

Applying Strategies to Sub-games

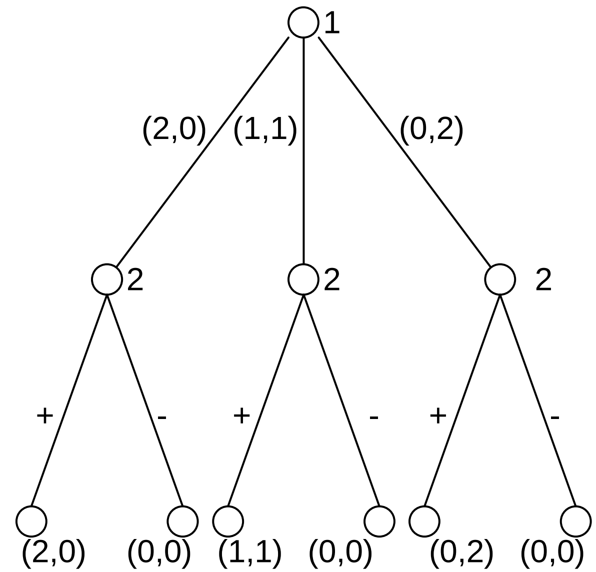
- If we have a strategy profile s^* for the game G and h is a history in G, then $s^*|_h$ is the strategy profile after history h, i.e., it is a strategy profile for *G(h)* derived from s^* by considering only the histories following *h.*
- For example, let ((out), (fight)) be a strategy profile for the entry game. Then ((),(fight)) is the strategy profile for the sub-game after player 1 played “in”.

Sub-game Perfect Equilibria

- A sub-game perfect equilibrium (SPE) of an extensive game with perfect information is a strategy profile s^* such that for all histories *h*, the strategies in $s^*|_h$ are optimal for all players.
- Note: ((out), (fight)) is not a SPE!
- Note: A SPE could also be defined as a strategy profile that induces a NE in every sub-game

Example: Distribution Game

- Two objects of the same kind shall be distributed to two players. Player 1 suggest a distribution, player 2 can accept (+) or reject (-). If she accepts, the objects are distributed as suggested by player 1. Otherwise nobody gets anything.
- NEs?
- SPEs?



- ((2,0),+xx) are NEs
- ((2,0),--x) are NEs
- ((1,1),-+x) are NEs
- ((0,1),--+) is a NE

Only

- ((2,0),+++) is a SPE
- ((1,1),-++) is a SPE

Existence of SPEs

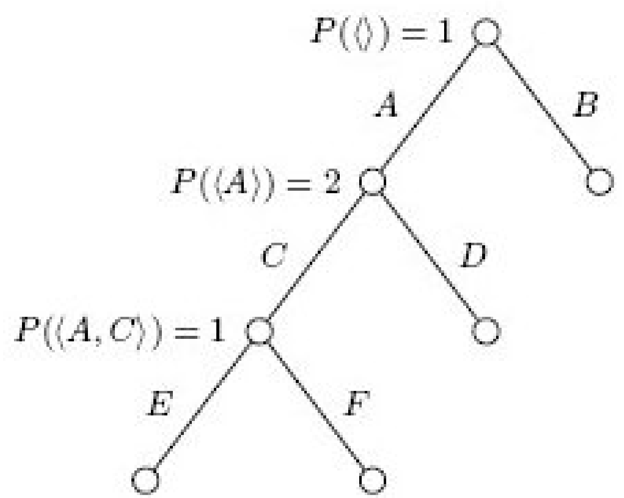
- Infinite games may not have a SPE
  - Consider the 1-player game with actions [0,1) and payoff $u_1(a) = a$.
- If a game does not have a finite horizon, then it may not possess an SPE:
  - Consider the 1-player game with infinite histories such that the infinite histories get a payoff of 0 and all finite prefixes extended by a termination action get a payoff that is proportional to their length.

Finite Games Always Have a SPE

- Length of a sub-game = length of longest history
- Use backward induction
  - Find the optimal play for all sub-games of length 1
  - Then find the optimal play for all sub-games of length 2 (by using the above results)
  - ….
  - until length n = length of game
  - ➢ game has an SPE
- SPE is not necessarily unique – agent my be indifferent about some outcomes
- All SPEs can be found this way!

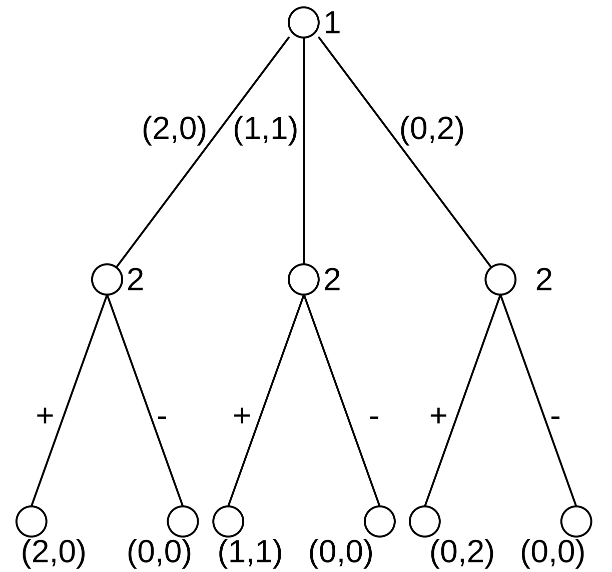
Strategies and Plans of Action

- Strategies contain decisions for unreachable situations!
- Why should player 1 worry about the choice after A,C if he will play B?
- Can be thought of as
  - what player 2 beliefs about player 1
  - what will happen if by mistake player 1 chooses A
  - Player 1 actually would play



The Distribution Game - again

- Now it is easy to find all SPEs
- Compute optimal actions for player 2
- Based on the results, consider actions of player 1

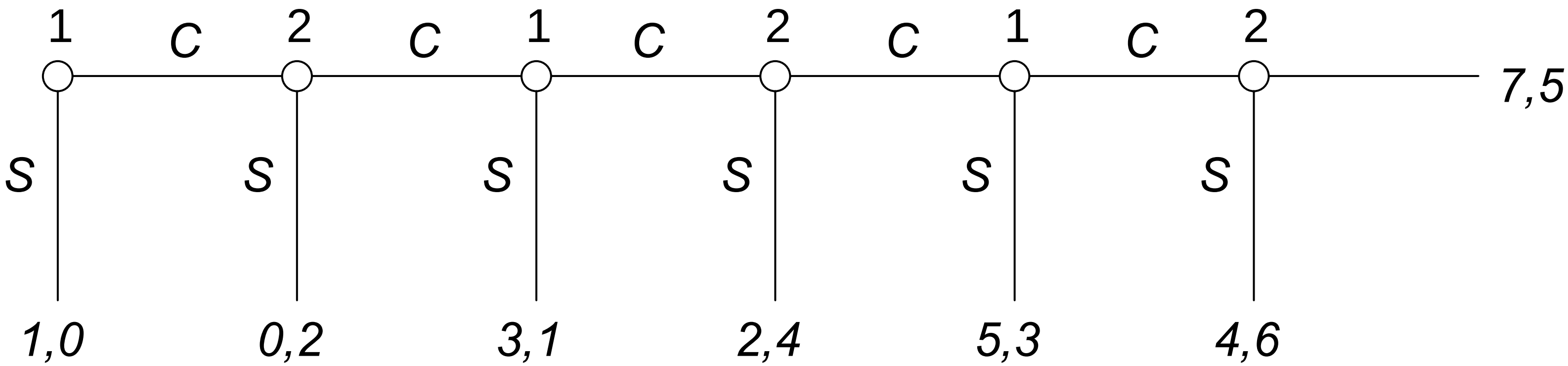


Another Example:
The Chain Store Game

- If we play the entry game for *k* periods and add up the payoff from each period, what will be the SPEs?
- By backward induction, the only SPE is the one, where in every period (in, give_in) is selected
- However, for the incumbent, it could be better to play sometimes fight in order to "build up a reputation" of being aggressive.

Yet Another Example: The Centipede Game

- The players move alternately
- Each prefers to stop in his move over the other player stopping in the next move
- However, if it is not stopped in these two periods, this is even better
- What is the SPE?



Relationship to *Minimax*

- Similarities to *Minimax*
  - solving the game by searching the game tree bottom-up, choosing the optimal move at each node and propagating values upwards
- Differences
  - More than two players are possible in the backward-induction case
  - Not just one number, but an entire payoff profile
- So, is *Minimax* just a special case?

Possible Extensions

- One could add random moves to extensive games. Then there is a special player which chooses its actions randomly
  - SPEs still exist and can be found by backward induction. However, now the expected utility has to be optimized
- One could add simultaneous moves, that the players can sometimes make moves in parallel
  - SPEs might not exist anymore (simple argument!)
- One could add “imperfect information”: The players are not always informed about the moves other players have made.

Conclusions

- Extensive games model games in which more than one simultaneous move is allowed
- The notion of Nash equilibrium has to be refined in order to exclude implausible equilibria – those with empty threats
- Sub-game perfect equlibria capture this notion
- In finite games, SPEs always exist
- All SPEs can be found by using backward induction
- Backward induction can be seen as a generalization of the Minimax algorithm
- A number of plausible extenions are possible: simulataneous moves, random moves, imperfect information