An Introduction to Game Theory
Part IV:
Games with Imperfect Information

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Motivation

- So far, we assumed that all players have perfect knowledge about the preferences (the payoff function) of the other players
- Often unrealistic
- For example, in auctions people are not sure about the valuations of the others
  - what to do in a sealed bid auction?

Example

- Let’s assume the *BoS* game, where player 1 is not sure, whether player 2 wants to *meet* her to *avoid* her,
- She assumes a probability of 0.5 for each case.
- Player 2 knows the preferences of player 1

Example (cont.)

	Bach	Stra-vinsky
Bach	2,1	0,0
Stra-vinsky	0,0	1,2

Prob. 0.5

	Bach	Stra-vinsky
Bach	2,0	0,2
Stra-vinsky	0,1	1,0

Prob. 0.5

What is the Payoff?

- Player 1 views player 2 as being one of two possible *types*
- Each of these types may make an independent decision
- So, the friendly player 2 may choose B and the unfriendly one S: (B,S)
- Expected payoff when player 1 plays B:

  $0.5 \times 2 + 0.5 \times 0 = 1$

Expected Payoffs & Nash Equilibrium

	(B,B)	(B,S)	(S,B)	(S,S)
B	2 (1,0)	1 (1,2)	1 (0,0)	0 (0,2)
S	0 (0,1)	0.5 (0,0)	0.5 (2,1)	1 (2,0)

- A Nash equilibrium in pure strategies is a triple (x,(y,z)) of actions such that:
  - the action x of player 1 is optimal given the actions (y,z) of both types of player 2 and the belief about the state
  - the actions y and z of each type of player 2 are optimal given the action x of player 1

Nash Equilibria?

	(B,B)	(B,S)	(S,B)	(S,S)
B	2 (1,0)	1 (1,2)	1 (0,0)	0 (0,2)
S	0 (0,1)	0.5 (0,0)	0.5 (2,1)	1 (2,0)

- Is there a Nash equilibrium?
  - Yes: B, (B,S)
- Is there a NE where player 1 plays S?
  - No

Formalization: States and Signals

- There are states, which completely determine the preferences / payoff functions
  - In our example: *friendly* and *unfriendly*
- Before the game starts, each player receives a signal that tells her something about the state
  - In our example:
    - Player 2 receives a states, which type she is
    - Player 1 gets no information about the state and has only her beliefs about probabilities.
- Although, the actions for non-realized types of player 2 are irrelevant for player 2, they are necessary for player 1 (and therefore also for player 2) when deliberating about possible action profiles and their payoffs

General Bayesian Games

- A Bayesian game consists of
  - a set of players $N = \{1, \ldots, n\}$
  - a set of states $\Omega = \{\omega_1, \ldots, \omega_k\}$
- and for each player i
  - a set of actions A_i
  - a set of signals T_i and a signal function $\tau_i: \Omega \rightarrow T_i$
  - for each signal a belief about the possible states (a probability distribution over the states associated with the signal) $Pr(\omega \mid t_i)$
  - a payoff function $u_i(a,\omega)$ over pairs of action profiles and states, where the expected value for a_i represents the preferences:

    $\sum_{\omega \in \Omega} Pr(\omega \mid t_i)\ u_i((a_i,\hat{a}_{-i}(\omega)),\omega)$

    with $\hat{a}_i(\omega)$ denoting the choice by i when she has received the signal $\tau_i(\omega)$

Example: BoS with Uncertainty

- Players: {1, 2}
- States: {friendly, unfriendly}
- Actions: {B, S}
- Signals: *T={a,b,c}*
  - $\tau_1(\omega_i) = a$ *for* $i=1,2$
  - $\tau_2(friendly) = b$, $\tau_2(unfriendly) = c$,
- Beliefs*:*
  - *Pr(friendly | a) = 0.5, Pr(unfriendly | a) = 0.5*
  - *Pr(friendly | b) = 1, Pr(friendly | b) = 0*
  - *Pr(friendly | c) = 0, Pr(friendly | c) = 1*
- Payoffs*:* As in the left and right tables *on the slide*

Example: Information can hurt

- In single-person games, knowledge can never hurt, but here it can!
- Two players, both don't know which state und consider both states ω_1 *and* ω_2 as equally probable (0.5)
- Note: Preferences of player 1 are known, while the preferences of player 2 are unknown (to both!)

ω1	L	M	R
T	3,2	3,0	3,3
B	6,6	0,0	0,9

ω2	L	M	R
T	3,2	3,3	3,0
B	6,6	0,9	0,0

Example (cont.)

- Player 2's unique best response is: L
- For this reason, player 1 will play B
- Payoff: 6,6 – only NE, even when mixed strategies!
- When player 2 can distinguish the states, R and M are dominating actions
- (T,(R,M)) is the unique NE

ω1	L	M	R
T	3,2	3,0	3,3
B	6,6	0,0	0,9

ω2	L	M	R
T	3,2	3,3	3,0
B	6,6	0,9	0,0

Incentives and Uncertain Knowledge May Lead to Suboptimal Solutions

- $\tau_1(\alpha) = a,\ \tau_1(\beta) = b,\ \tau_1(\gamma) = b$
  - $Pr(\alpha|a) = 1$
  - $Pr(\beta|b) = 0.75$, $Pr(\gamma|b) = 0.25$
- $\tau_2(\alpha) = c,\ \tau_2(\beta) = c,\ \tau_2(\gamma) = d$
  - $Pr(\alpha|c) = 0.75$, $Pr(\beta|c) = 0.25$
  - $Pr(\gamma|d) = 1$
- In state γ, there are 2 NEs
- In state γ, player 2 knows her preferences, but player 1 does not know that!
- The incentive for player 1 to play R in state α „infects“ the game and only (R,R),(R,R) is an NE

α	L	R
L	2,2	0,0
R	3,0	1,1

β & γ	L	R
L	2,2	0,0
R	0,0	1,1

The Infection

- Player 1 must play R when receiving signal a (= state α)!
- Player 2 will therefore never play L when receiving c (= α or β)
- For this reason, player 1 will never play L when receiving b (= β or γ)
- Therefore player 2 will also play R when receiving d (= γ)
- Therefore the unique NE is ((R,R),(R,R))!

α	L	R
L	2,2	0,0
R	3,0	1,1

β & γ	L	R
L	2,2	0,0
R	0,0	1,1

*τ1(*α*) = a, τ1(*β*) = b, τ1(*γ*) = b*
*Pr(*α|a) = 1
*Pr(*β|b) = 0.75, *Pr(*γ |b) = 0.25

*τ2(*α*) = c, τ2(*β*) = c, τ2(*γ*) = d*
*Pr(*α|c) = 0.75, *Pr(*β|c) = 0.25
*Pr(*γ|d) = 1

Auctions with Imperfect Information

- Players: N = {1, …, n}
- States: the set of all profiles of valuations $(v_1,\ldots,v_n)$, where $0 \leq v_i \leq v_{max}$
- Actions: Set of possible bids
- Signals: The set of the player *i*'s valuation $\tau_i(v_1,\ldots,v_n) = v_i$
- Beliefs: $F(v)$ is the probability that the other bidder values of the object is *at most v, i.e.,* $F(v_1)x\ldots xF(v_{i-1})xF(v_{i+1})x\ldots xF(v_n)$ is the probability, that all other players *j* value the object at most v_j
- Payoff: $u_i(b,(v_1,\ldots,v_n)) = (v_i - P(b))/m$ if $b_j \leq b$ for all $i \neq j$ and $b_j = b$ for *m* players and $P(b)$ being the *price function:*
  - *P(b)* the highest bid = first price auction
  - *P(b)* the second highest bid = second price auction

Private and Common Values

- If the valuations are *private,* that is each one cares only about the his one appreciation (e.g., in art),
  - valuations are completely independent
  - one does not gain information when people submit public bids
- In an auction with *common valuations,* which means that the players share the value system but may be unsure about the real value (antiques, technical devices, exploration rights),
  - valuations are not independent
  - one might gain information from other players bids
- Here we consider private values

Second Price Sealed Bid Auction

- *P(b)* is what the second highest bid was
- As in the perfect information case (see exercise):
  - It is a weakly dominating action to bid ones own valuation v_i
  - There exist other, non-efficient, equlibria

First Price Sealed Bid Auction

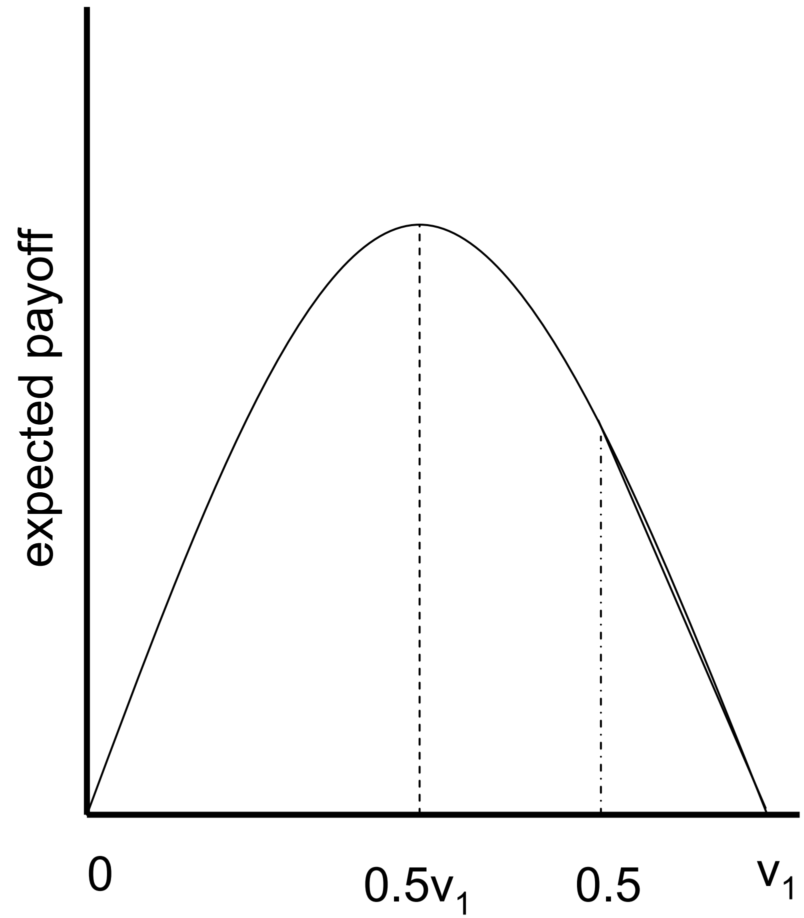
- A bid of v_i weakly dominates any bid higher than v_i
- A bid of v_i does not weakly dominate a bid lower than v_i
- A bid lower than v_i weakly dominates v_i
- NE probably at a point below v_i
- General analysis is quite involved
- Simplifications:
  - only 2 players
  - $v_{max} = 1$
  - uniform distribution of valuations, i.e., $F(v) = v$

First Price Sealed Bid Auction (2)

- Let $B_i(v)$ the bid of type v for player i.
- *Claim*: Under the mentioned conditions, the game has a NE for $B_i(v) = v/2$.
- Assume that player 2 bids this way, then as far as player 1 is concerned, player 2's bids are uniformly distributed between 0 and 0.5.
- Thus, if player 1 bids $b_1 > 0.5$, she wins. Otherwise, the probability that she wins is $F(2b_1)$
- The payoff is
  - $v_1 - b_1$ if $b_1 > 0.5$
  - $2b_1 (v_1 - b_1) = 2b_1v_1 - 2b_1^2$ if $0 \leq b_1 \leq 0.5$

First Price Sealed Bid Auction (3)

- In other words, $0.5v_1$ is the best response to $B_2(v)=v/2$ for player 1.
- Since the players are symmetric, this also holds for player 2
- Hence, this is a NE
- In general, for m players, the NE is $Bi(v)=v/m$ for m players
- Can also be shown for general distributions



Conclusion

- If the players are not fully informed about there own and others utilities, we have imperfect information
- The technical tool to model this situation are Bayesian games
- New concepts are states, signals, beliefs and expected utilities over the believed distributions over states
- Being informed can hurt!
- Auctions are more complicated in the imperfect information case, but can still be solved.