Social Robotics

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Non-parametric Tests

Overview

- Wilcoxon signed-rank test
- Wilcoxon rank-sum test (aka Mann-Whitney test)
- Kruskal-Wallis test

Ranks

- Ranks are natural numbers starting with 1, which get assigned to scores sorted in increasing order.
- Ranks can be assigned to any data which is at least ordinal.
- Ranks are robust against outliers (because ranks are used instead of the actual data).

Example

- Data: 0, 7, 3; Rank: 1, 3, 2
- Data: -100, 99, 98; Rank: 1, 3, 2
- Data: d, a, b; Rank: 3, 1, 2

Ranks: Ties

- In case of ties, the average rank is assigned to the whole group of scores that constitutes the tie.

Example

- Data: 1, 6, 4, 4, 2, 2, 2
- Rank: 1, 7, 5.5, 5.5, 3, 3, 3

Wilcoxon Signed-Rank Test: Motivation

- Likert scales are a popular means of measurement.
- Likert scales in most cases have no interval-scale reading.

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale ”never ever!” (1), “maybe” (2), “yes, sure!” (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
- Difference: -1, -1, -1, 0, +2
- Differences without 0: -1, -1, -1, +2
- Ranks: 2, 2, 2, 4

Wilcoxon Signed-Rank Test: Example Continued

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale ”never ever!” (1), “maybe” (2), “yes, sure!” (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
- Difference: -1, -1, -1, 0, +2; Without 0: -1, -1, -1, +2
- Ranks: 2, 2, 2, 4

- Let $V = \sum_i^n Z_i R_i$ be the sum of the positive ranks ($Z_i = 1$ if difference i is positive, and $Z_i = 0$ else).
- In the example $V = 4$. Well, so what?

Towards the Null Hypothesis

- Imagine two paired samples and consider their rank differences.
- Consider $V = \sum_i^n Z_i R_i$. What could happen?
  1. Case $V = 0$: All the rank differences are negative.
  2. Case $V = \sum_i^n R_i = \frac{n(n+1)}{2}$: All rank differences are positive.
  3. Else: V ranges between 0 and $\frac{n(n+1)}{2}$.
- If the groups do not differ (H_0), then 50% of the differences should be below 0 and 50% above. This is like saying that the median of the difference is 0. And in that case, V should be close to $\frac{\frac{n(n+1)}{2}}{2} = \frac{n(n+1)}{4}$.
- Hence, we will test $H_0 : Mdn = 0$ against its alternatives, and we will do that by using V.

Wilcoxon Signed-Rank Test

- The nice thing about V is that (for $n > 25$) its distribution is well approximated by a normal distribution $\mathcal{N}(\mu_V, \sigma_V)$ with
  - $\mu_V = \frac{n(n+1)}{4}$
  - $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24}}$

Wilcoxon Signed-Rank Test: Mean

- The nice thing about V is that (for $n > 25$) its distribution under H_0 is well approximated by a normal distribution $\mathcal{N}(\mu_V, \sigma_V)$ with
  - $\mu_V = \frac{n(n+1)}{4}$
  - $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24}}$
- Proof (Mean): We already came to this conclusion earlier on Slide 8.

Wilcoxon Signed-Rank Test: Mean

- The nice thing about V is that (for $n > 25$) its distribution is well approximated by a normal distribution $\mathcal{N}(\mu_V, \sigma_V)$ with
  - $\mu_V = \frac{n(n+1)}{4}$
  - $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24}}$
- Proof (Variance)
  - First, we define $V' = \sum_i^n V_i'$ with

    $$V_i' = \begin{cases} 0 & \textit{with probability } 0.5 \\ i & \textit{with probability } 0.5 \end{cases}$$
  - (V' has the same distribution as V, because, for every rank, it either belongs to the sum of V or not with probability 0.5.)
  - $Var(V) = Var(V') = \sum_i^n Var(V_i')$ (independence of V_i').
  - $Var(V_i') = E(V_i'^2) - E(V_i')^2 = (0^2\frac{1}{2} + i^2\frac{1}{2}) - (\frac{1}{2}i)^2 = \frac{i^2}{4}$
  - $Var(V) = \sum_i^n Var(V_i) = \sum_i^n \frac{i^2}{4} = \frac{n(n+1)(2n+1)}{24}$.

Wilcoxon Signed-Rank Test: Example Again

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale "never ever!" (1), "maybe" (2), "yes, sure!" (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
- Difference: -1, -1, -1, 0, +2; Without 0: -1, -1, -1, +2
- Ranks: 2, 2, 2, 4
- $V = 4$, $\mu_v = 4(4+1)/4 = 5$, $\sigma_V = \sqrt{4(4+1)(2 \times 4+1)/24}$
- $z = \frac{V - \mu_v}{\sigma_V} = (4-5)/2.74 = -0.365$
- $p = P(z \leq -0.365) + 1 - P(z \leq 0.365) = 0.715$

Comparison to Paired t-Test

Example: t-Test

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots before and after they have watched a Sci-Fi movie. As a measurement instrument, a 3-Point Likert-Scale ”never ever!” (1), “maybe” (2), “yes, sure!” (3) was used.

- Before: 1, 2, 2, 3, 3; After: 2, 3, 3, 3, 1
- Difference: -1, -1, -1, 0, +2
- $\overline{D} = 0.20$, $s_D = 1.30$, $n = 5$
- $t = \sqrt{5} \times 0.20/1.30 = 0.344$, $df = 4$
- $p = 0.748$

Wilcoxon Rank-Sum Test: Motivation

Example

Five participants are asked to rate their belief in the possibility that humans will one day be the slaves of robots after they have watched the Sci-Fi movie M1, and five participants rate their belief after watching Sci-Fi movie M2. As a measurement instrument, a 3-Point Likert-Scale ”never ever!” (1), “maybe” (2), “yes, sure!” (3) was used.

- M1: 1, 1, 2, 2, 2; M2: 2, 3, 3, 3, 2
- H_0: The two groups are equal. I.e., they stem from a distribution of equal median.
- Reject H_0 or not?

Wilcoxon Rank-Sum Test: General Setting

- First, all scores are ranked together.
- First group's rank sum: $R_1 = \sum_{i=1}^{n_1} r_{1,i}$
- Second group's rank sum: $R_2 = \sum_{i=1}^{n_2} r_{2,i}$
- First group's W: $W_1 = R_1 - \sum_{i=1}^{n_1} i = R_1 - \frac{n_1(n_1+1)}{2}$
- Second group's W: $W_2 = R_2 - \sum_{i=1}^{n_2} i = R_2 - \frac{n_2(n_2+1)}{2}$
- $W_1 + W_2 = R_1 - \frac{n_1(n_1+1)}{2} + R_2 - \frac{n_2(n_2+1)}{2} = n_1 n_2$
- Note: The Wilcoxon Rank-Sum Test is also known as Mann-Whitney U-Test, and W is also called U. There are various ways of defining W (resp. U), which are all equal! R uses the statistics W the way shown above.

Wilcoxon Rank-Sum Test: Distribution of W

- For larger samples ($n_1 > 10, n_2 > 10$), $W \sim \mathcal{N}(\mu_W, \sigma_W)$:
  - $\mu_W = \frac{n_1 n_2}{2}$
  - $\sigma_W = \sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}}$
  - Also see simulation in `lecture10.Rmd`.
- Again, we can calculate z-values to decide whether or not W is extreme, i.e., whether or not to reject H_0.

Wilcoxon Rank-Sum Test: Example Continued

Example

- M1: 1, 1, 2, 2, 2
- M2: 2, 3, 3, 3, 2

- All Scores: 1, 1, 2, 2, 2, 2, 3, 3, 3, 2
- Ranks: 1.5, 1.5, 5, 5, 5, 5, 9, 9, 9, 5
- $R_1 = 18$, $W = 18 - 15 = 3$
- $z = \frac{3 - \frac{(5\times5)}{2}}{\sqrt{\frac{5\times5(5+5+1)}{12}}} = -2.298$
- $p = P(z \leq -2.298) + 1 - P(z \leq 2.298) = 0.022$

Wilcoxon Rank-Sum Test vs. t-Test

Example

- M1: 1, 1, 2, 2, 2
- M2: 2, 3, 3, 3, 2

- $\overline{X}_1 = 1.6$, $\overline{X}_2 = 2.6$, $s_1^2 = 0.3$, $s_2^2 = 0.3$, $n = 5$, $df = 8$
- $t = \sqrt{n} \times \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{s_1^2 + s_2^2}} = \sqrt{5} \times \frac{1.6 - 2.6}{\sqrt{0.3+0.3}} = -2.887$
- $p = P(t \leq -2.887) + 1 - P(t \leq 2.887) = 0.020$
- For a simulation comparing Wilcoxon and t-Test see `lecture11.Rmd`.

Kruskal-Wallis Test: Setting

- Also for rank-based methods, there is an analog to ANOVA that can cope with more than two groups: Kruskal-Wallis Test. As for ANOVA, H_0 reads "There is no difference between the groups".
- First, the scores of all groups are ranked together (like for Wilcoxon Rank-Sum Test).
- The test statistics is called H:
  - $H = (N-1)\frac{\sum_i^p n_i(\bar{r}_i - \bar{r})^2}{\sum_i^p \sum_j^{n_i} (r_{ij} - \bar{r})^2}$, with $N = \sum_i^p n_i$, $\bar{r}_i = \frac{\sum_j^{n_i} r_{ij}}{n_i}$, $\bar{r} = \frac{N+1}{2}$
  - H can be simplified to $H = \frac{12}{N(N+1)} \sum_i^p n_i \bar{r}_i^2 - 3(N+1)$
- $H \sim \chi^2_{p-1}$, with p being the number of groups.

Kruskal-Wallis Test: Example

Example

- M1: 1, 1, 2, 2, 2; Ranks: 2.5, 2.5, 8.5, 8.5, 8.5
- M2: 2, 3, 3, 3, 2; Ranks: 8.5, 14, 14, 14, 8.5
- M3: 1, 2, 2, 1, 2; Ranks: 2.5, 8.5, 8.5, 2.5, 8.5

- $\bar{r}_1 = 6.1$, $\bar{r}_2 = 11.8$, $\bar{r}_3 = 6.1$, $N = 15$, $\bar{r} = (15+1)/2 = 8$
- $H = \frac{12}{15\times16} \times 5(37.21 + 139.24 + 37.21) - 3 \times 16 = 5.41$
- $p = 1 - P(\chi^2 \leq 5.41) = 0.067$
- R will report different values, see next slide to learn why.

Ties call for Corrections

If there are long ties (i.e., a lot of scores getting the same rank), the variance of the statistics become smaller and thus some corrections have to be applied.

- The V-statistics's standard deviation becomes:
  - $\sigma_V = \sqrt{\frac{n(n+1)(2n+1)}{24} - \sum_i^k \frac{t_i^3 - t_i}{48}}$ (cf., slide 9)
- The W-statistics's standard deviation becomes:
  - $\sigma_W = \sqrt{\frac{n_1 n_2}{12}\left((n_1 + n_2 + 1) - \sum_i^k \frac{t_i^3 - t_i}{(n_1+n_2)(n_1+n_2-1)}\right)}$ (cf., slide 16)
- And the H-statistics can be corrected by dividing H by the term $corr = 1 - \frac{\sum_i^k (t_i^3 - t_i)}{N^3 - N}$
  - In the example: $corr = 1 - \frac{(4^3-4)+(8^3-8)+(3^3-3)}{(15^3-15)}$
  - The corrected H value then is $H_{corr} = 6.56$
- Because all this is rather tedious, you are allowed to skip these corrections in your assignments (also in the exam).

Current State of our Toolkit

- Categorical Scale
  - χ^2-statistics (χ^2-distributed)
- Interval Scale
  - Variance known: z-statistics (normally distributed)
  - Variance unknown (but equal): t-statistics (Student's t distribution), F-statistics (F-distributed)
- Ordinal Scale
  - W-, V-statistics (both normally distributed), H-statistics (χ^2-distributed)

What comes next

- We started out defining four types of hypotheses
  1. Directional difference hypotheses
  2. Undirectional difference hypotheses
  3. Directional relationship hypotheses
  4. Undirectional relationship hypotheses
- We can so far only deal with (1) and (2). This is going to be fixed during the remaining lectures. Stay tuned!

Sketches

Intentionally left blank :-)