Social Robotics

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Inferential Statistics (Intro)

Current State

- You know how to look at your data.
- You know how to present your data.
- You got a first impression how to judge a data point as *extreme* or *usual* using IQR or z-Score.

The Problem

We face the problem that we want to investigate, whether some universally quantified statement holds, while we only have access to a subset of the overall population of entities the statement is quantifying over. This subset of the population we have access to is called the sample.

$\Rightarrow$Inferential statistics is about what we can reasonably say about the population given a sample.

Statistics vs. Parameters

	statistics	parameter
Mean	$\overline{X} = \frac{1}{N}\sum_i^N X_i$	$\mu = \frac{1}{N^*}\sum_i^{N^*} X_i$
Variance	$s^2_{biased} = \frac{1}{N}\sum_i^N (X_i - \overline{X})^2$ $s^2_{unbiased} = \frac{1}{N-1}\sum_i^N (X_i - \overline{X})^2$	$\sigma^2 = \frac{1}{N^*}\sum_i^{N^*} (X_i - \mu)^2$
Standard Deviation	$\sqrt{s^2}$	$\sqrt{\sigma^2}$

Sampling Distribution of the Sample Mean

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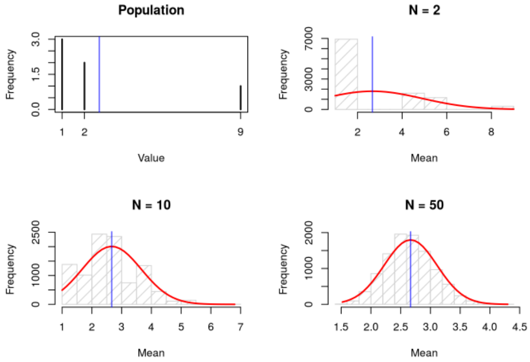
The Gist

The sample mean will be approximately normally distributed for large sample sizes, regardless of the distribution from which we are sampling.

Evidence by Simulation



- Blue lines: Population mean μ.
- Grey Bars: Frequency of sampled means
- Red Gaussian: $\mathcal{N}(\mu, \frac{\sigma^2}{N})$

Mean of the Sampling Distribution of the Sample Mean

Let $X_1, \ldots, X_N$ be N independently drawn observations from a distribution with mean μ and variance σ^2. Thus, $E[X_i] = \mu$ for all i. Let's derive $E[\overline{X}]$, which we call the mean of the sampling distribution of the sample mean (also written as $\mu_{\overline{X}}$):

$$E[\overline{X}] = E[\frac{1}{N}\sum_i^N X_i] = \frac{1}{N}E[\sum_i^N X_i] = \frac{1}{N}\sum_i^N E[X_i] = \frac{1}{N}N\mu = \mu$$

Variance of the Sampling Distribution of the Sample Mean

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Let $X_1, \ldots, X_N$ be N independently drawn observations from a distribution with mean μ and variance σ^2. Thus, $Var[X_i] = \sigma^2$ for all i. Let's derive $Var[\overline{X}]$, which we call the variance of the sampling distribution of the sample mean (also written as $\sigma^2_{\overline{X}}$):

$$Var[\overline{X}] = Var[\frac{1}{N}\sum_i^N X_i] = (\frac{1}{N})^2 Var[\sum_i^N X_i] = (\frac{1}{N})^2 \sum_i^N Var[x_i] = (\frac{1}{N})^2 N\sigma^2 = \frac{\sigma^2}{N}$$

- Hence, the standard deviation of the sampling distribution of the sample mean is $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{N}}$.
- $\sigma_{\overline{X}}$ is also called the Standard Error.

Summary: Sampling Distribution of the Sample Mean

$$\overline{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N})$$

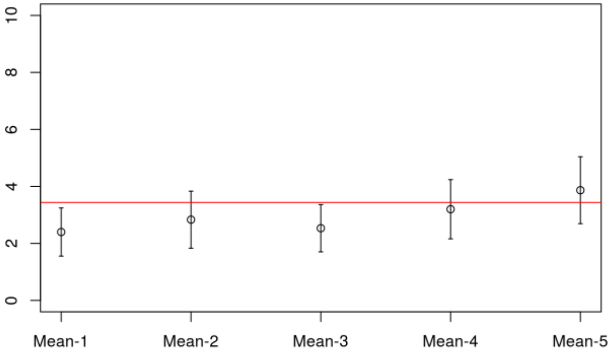
Application: Confidence Intervals I

- Suppose we know the population mean μ and standard deviation σ.
- Can we find boundaries within which we believe the mean of a sample of size N will fall with 95% probability?
- We know how our sample means are distributed, viz., $\overline{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N})$
  - The lower boundary $\overline{X}_{low}$ will be 1.96 standard errors below μ, and the upper boundary $\overline{X}_{up}$ will be 1.96 standard errors above μ.
  - $\mu - \overline{X}_{low} = 1.96 \times \frac{\sigma}{\sqrt{N}} \Rightarrow \overline{X}_{low} = \mu - 1.96 \times \frac{\sigma}{\sqrt{N}}$
  - $\overline{X}_{up} - \mu = 1.96 \times \frac{\sigma}{\sqrt{N}} \Rightarrow \overline{X}_{up} = \mu + 1.96 \times \frac{\sigma}{\sqrt{N}}$

Application: Confidence Intervals II

- Suppose we have collected some sample of size N, and we have computed the $\overline{X}$- and s^2-statistics.
- Can we find boundaries within which we believe the population mean μ will fall with 95% probability?
- We just look from the "sample's perspective".
- In need of parameters, we estimate $\mathcal{N}(\mu, \frac{\sigma^2}{N})$ by $\mathcal{N}(\overline{X}, \frac{s^2}{N})$ (which is okay, if $N > 30$).
  - The lower boundary X_{low} will be 1.96 standard errors below $\overline{X}$, and the upper boundary X_{up} will be 1.96 standard errors above $\overline{X}$.
  - $\overline{X} - X_{low} = 1.96 \times \frac{s}{\sqrt{N}} \Rightarrow X_{low} = \overline{X} - 1.96 \times \frac{s}{\sqrt{N}}$
  - $X_{up} - \overline{X} = 1.96 \times \frac{s}{\sqrt{N}} \Rightarrow X_{up} = \overline{X} + 1.96 \times \frac{s}{\sqrt{N}}$

Means and Confidence Intervals



- Red line: Population mean μ
- Dots: Sampled Means
- Lines through dots: 95% confidence intervals

Reporting Confidence Intervals

Report

We recorded the number of interactions with our robot per day for nine days (N = 9). The number of interactions ranged from 35 to 150 ($\overline{X}$ = 65.11, s = 33.59, 95% CI [43.16, 87.05]).

- Remember the data 35, 50, 50, 50, 56, 60, 60, 75, 150.

Application: Hypothesis Testing

- The sample mean has a distribution that is normal (for sufficiently large sample sizes), even when we are sampling from a distribution that is not normal.
- This is useful, because given μ and σ, we can compute the probability that some sample of size N with mean $\overline{X}$ stems from that population!
- We already know how we can judge whether some value from a normal distribution is ‘usual’ or rather ‘extreme’: z-Scores!
- Hence, we can judge a sample mean as ‘usual’ or ‘extreme’ by computing its z-Score.
- Let’s see how we can use this for hypothesis testing!

Very First Hypothesis Test

Suppose you have been deploying a robot (Robo-One) in your museum. You have recorded the number of interaction for a very long time, such that you can assume the collected mean and variance of the number of interactions to be the population mean $\mu_0 = 40$ and standard deviation $\sigma_0 = 4$. You have now bought a fancy new version of the robot, viz., Robo-Two. Your Hypothesis is that Robo-Two will generate much more interactions compared to Robo-One.

- Hypothesis H_1: Robo-Two generates more interactions than Robo-One.
- H_1 is of type (difference, directional)
- Can be written as $H_1 : \mu > \mu_0$, i.e., the population mean for interactions with Robo-Two (μ) is bigger than the population mean for interactions with Robo-One (μ_0), i.e., people generally interact more with Robo-Two than with Robo-One.

Very First Hypothesis Test

- The trick of inferential statistics is to first assume that the negation of H_1 is the case, which is called the Null-Hypothesis, written H_0.
- Then, we collect the data (viz., our sample)
- Subsequently, we show that our sample is so unlikely under H_0 that we are allowed to reject H_0 in favor of H_1.
  - In the example: $H_1 : \mu > \mu_0$, $H_0 : \mu \leq \mu_0$.

Very First Hypothesis Test

- Next, we record the number of interactions of Robo-Two for 16 days ($N = 16$), and we find a mean $\overline{X} = 42$.
- Given the population mean and standard deviation $\mu_0 = 40$ and $\sigma_0 = 4$, we know that the sampling distribution of the sample mean is $\mathcal{N}(40, \frac{16}{16})$.
- We compute the z-Score to assess how far our sample mean 42 is from the mean of the sampling distribution of the sample mean, 40: $z = (42 - 40)/\frac{4}{4} = (42 - 40) = 2$.

Very First Hypothesis Test

- Thus, observing a sampling mean of at least 42 under the assumption that the population mean is $\mu_0 = 40$ and the population standard deviation is $\sigma_0 = 4$ is as probable as $P(z \geq 2) = 1 - P(z < 2) = 0.0228$.
- Things will become even worse if we consider population means smaller than μ_0. Therefore, if we assume a significance level of $\alpha = 0.05$, we have reason to reject H_0 in favor of H_1.

Very First Hypothesis Test

Report

The number of interactions with Robo-Two is significantly higher than the number of interactions with Robo-One ($z = 2.0, p = 0.0228$).

- Because the hypothesis was directional, we checked if the z-Score of $\overline{X}$ was $z_{.95} = 1.65$ or higher. This is called a one-tailed test. The p-Value is just the probability $P(z \geq 2.0) = 0.0228$. This is below the significance level $\alpha = 0.05$.

Second Hypothesis Test

- This time, our H_1 hypothesis was that there is a difference between Robo-One and Robo-Two: $H_1 : \mu \neq \mu_0$.
- The null-hypothesis then is $H_0 : \mu = \mu_0$.
- We will reject H_0, if μ is too low or too high. Thus, we split our 5% significance level into two (2.5% at the lower end, and 2.5% at the higher end).
- We thus check if the z-Value is below $z_{.025} = -1.96$ or above $z_{.975} = 1.96$. This is a two-tailed test.
- As our z-Score was 2, we will also reject H_0 this time.

Very First Hypothesis Test

Report

The number of interactions with Robo-Two and with Robo-One differ significantly ($z = 2.0, p = 0.044$).

- Because the hypothesis was non-directional, we compute the probability to observe a z-Score at least as extreme as 2.0 (in both directions). The probability is thus $P(z \geq 2.0) + P(z \leq -2.0) = 0.0228 + 0.0228 = 0.0456$. This is below the significance level $\alpha = 0.05$.

Third Hypothesis Test

- This time, our H_1 hypothesis was that there there will be less interactions with Robo-Two than with Robo-One: $H_1 : \mu < \mu_0$.
- The null-hypothesis then is $H_0 : \mu \geq \mu_0$.
- We will reject H_0 if μ is too low. Thus, we test at the lower 5% tail, viz., if the z-Score is less or equal $z_{.05} = -1.65$.
- As our z-Score was 2, we will not reject H_0.

Very First Hypothesis Test

Report

The hypothesis H_1 stating that the number of interactions with Robo-Two will be less than with Robo-One was not supported ($z = 2.0, p = 0.9772$).

- This time we look only at the lower end, thus, we compute the probability $P(z \leq 2.0) = 0.9772$, which clearly is above the significance level $\alpha = 0.05$.

Type-I and Type-II Errors

- Our decisions to reject H_0 or not are based on probabilities! We see that our sample would be rather unusual if H_0 were true, thus we reject H_0. But it could be that we just had an unusual sample by chance. If we decide to reject H_0 although H_0 is actually true, then we commit a Type-I Error. Using the 5% significance level, we have a 5% chance per rejected H_0 hypothesis that we were wrong.
- If we instead reject H_1 although H_0 is wrong, then we commit a Type-II Error. This can happen, when there is an effect in the population, but our sample size was too small to detect that effect.

Final Note on z-Test

- Note that we have assumed that μ and σ^2 are known to us a-priori, or can be reasonably be approximated in case of a sufficiently big sample size.
- In many applications, we will not be able to enjoy this luxury.
- Therefore, we will learn about other test statistics, as well. But the main idea is the same, most of the time.

Sketches

Intentionally left blank :-)